Atomic Norm Minimization for Modal Analysis from Compressive Measurements

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I. Introduction

One analytical technique for assessing the health of a structure such as a building or bridge is to estimate its mode shapes and frequencies via vibrational data collected from the structure. A change in a structure's modal parameters could be indicative of damage. Due to the considerable time and expense required to perform manual inspections of physical structures, and the difficulty of repeating these inspections frequently, there is a growing interest in developing automated techniques for structural health monitoring (SHM) based on data collected in a wireless sensor network (WSN) [1], [2], [3].

In order to save energy and extend battery life, it is desirable to reduce the dimension of data that must be collected and transmitted in the WSN. In recent work [4], Park et al. provided a rigorous analysis of a singular value decomposition (SVD) based technique for estimating the structure's mode shapes in free vibration without damping. As a means of compression, this work considered both random sampling in time and multiplication by random matrices. While promising, the SVD-based algorithm requires orthogonality of the mode shapes and offers only approximate, not exact recovery.

Recently, atomic norm minimization (ANM) based approaches for line spectrum estimation have been shown to be efficient and powerful for exactly recovering unobserved samples and identifying off-grid frequencies in both single measurement vector (SMV) [5], [6] and multiple measurement vector (MMV) [7], [8] scenarios. In particular, theoretical guarantees have been established for random sampling in time when the sampling times are synchronous [7] and asynchronous [8] across the sensors. However, these guarantees assume randomness of the mode shapes, which is not physically plausible. Moreover, these guarantees suggest that sample complexity per sensor will *increase* as the the number of sensors increases, which is both undesirable and contrary to intuition.

In this work, we consider the modal analysis problem when data is compressed at each sensor via multiplication by a random matrix. We show that ANM can perfectly recover modal parameters even when the mode shapes are not orthogonal. We provide new theoretical analysis on the sample complexity of this scheme. In particular, our theory does not require randomness of the mode shapes, and it shows that the sample complexity per sensor will *decrease* as the the number of sensors increases. Our theory can be interpreted as an extension of the SMV treatment in [6] to the MMV scenario.

II. Problem Formulation

We consider the vector-valued analytical signal as in [4]

$$\boldsymbol{x}^\star(t) = \sum_{k=1}^{K} A_k \boldsymbol{\psi}_k^\star e^{j2\pi F_k t},$$

which is a superposition of K complex sinusoids at each sensor. Here, $A_k \in \mathbb{C}$, $F_k \in \mathbb{R}$ and $\boldsymbol{\psi}_k^\star \in \mathbb{C}^N$ are the complex amplitudes, frequencies and normalized mode shapes, respectively, and N is the number of sensors. Taking evenly spaced Nyquist samples with $T_s \le \frac{1}{2\max\{F_k\}}$ at times $t_1, t_2, \dots, t_M$, we can form an $M \times N$ data matrix $\mathbf{X}^\star = [\boldsymbol{x}(t_1) \cdots \boldsymbol{x}(t_M)]^\top$, which can be written as a superposition

$$\mathbf{X}^\star = \sum_{k=1}^{K} |A_k| \boldsymbol{a}(F_k) \boldsymbol{\psi}_k^\top, \tag{1}$$

where the atoms $\boldsymbol{a}(f) = [e^{j2\pi F t_1}, \cdots, e^{j2\pi F t_M}]^\top$, $\boldsymbol{\psi}_k = \boldsymbol{\psi}_k^\star e^{j\phi_k}$, and ϕ_k is the phase of the complex amplitude A_k.

III. Random Temporal Compression

Let $\boldsymbol{x}_n^\star$ be the n^{th} column of the data matrix $\mathbf{X}^\star$ (corresponding to the raw data at sensor n), and consider the compressive measurements

$$\boldsymbol{y}_n = \boldsymbol{\Phi}_n \boldsymbol{x}_n^\star, \quad n = 1, \dots, N.$$

Here, each $\boldsymbol{\Phi}_n \in \mathbb{R}^{M' \times M}$ is a Gaussian random matrix. From these measurements, we can recover the original data matrix $\mathbf{X}^\star$ and the modal parameters via the ANM program

$$\hat{\mathbf{X}} = \arg\min_{\mathbf{X}} \|\mathbf{X}\|_{\mathcal{A}} \;\; \text{s.t. } \boldsymbol{y}_n = \boldsymbol{\Phi}_n \boldsymbol{x}_n, \quad n = 1, \dots, N. \tag{2}$$

since only a few atoms are used to represent the data matrix in (1). Here, $\|\mathbf{X}\|_{\mathcal{A}}$ is the atomic norm induced by the atoms $\boldsymbol{a}(f)\boldsymbol{\psi}^\top$ [5].

Theorem III.1. *Let $\boldsymbol{\Phi} = [\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2, \cdots, \boldsymbol{\Phi}_N] \in \mathbb{R}^{M' \times MN}$ be a random matrix with i.i.d. zero-mean, unit variance Gaussian entries. Assume that the true frequencies satisfy the minimum separation condition in [5]. Then, there exists some constant C such that $\mathbf{X}^\star$ is the unique optimal solution of (2) with probability at least $1 - \delta$ if*

$$M' \ge \max\left\{ \frac{8}{N} \log\left(\frac{1}{\delta}\right) + 2, CK\log(M) \right\}, \tag{3}$$

and we can exactly recover the frequencies and mode shapes up to a phase ambiguity.

This result, which does not impose any randomness assumption on the mode shapes, indicates that the number of measurements needed from each sensor scales with $K\log(M)$ and that the probability of exact recovery increases as the number of sensors N increases. For more details and proofs, please refer to our manuscript [9].

Simulation results are presented in Figures 1, 2 and 3 to verify our theory. For joint recovery (solid lines), it can be seen in Figures 1 and 2 that the probability of successful recovery does increase as the number of sensors increases; alternatively, the number of measurements needed for perfect recovery will decrease as we have more sensors. We also show the recovery performance using ANM separately at each sensor (dashed lines), to illustrate the advantage of joint recovery. Finally, it is shown in Figure 3 that the sample complexity M' scales linearly with the number of active modes K as is indicated in Theorem III.1.

At the workshop, we will discuss this result in more depth as well as survey the different compression strategies and compare with results from [7], [8].

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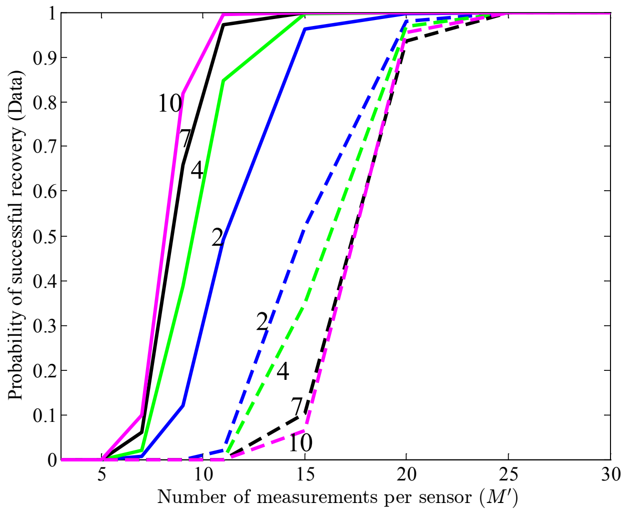

Fig. 1. Random temporal compression: recovery of data matrix via joint ANM (solid lines) and separate ANM (dashed lines). We take $M' = [3, 5, 7, 9, 11, 15, 20, 25, 30]$ measurements per sensor and perform 500 trials with random mode shapes. The true frequencies are set as $F_1 = 2$, $F_2 = 10$, $F_3 = 20$ Hz, which are well separated. The amplitudes are set as $A_1 = 1$, $A_2 = 2$, $A_3 = 3$. The raw data consists of $M = 100$ uniform samples taken with a sampling rate of 100 Hz before compression. For separate recovery, we recover each column of the data matrix separately as in the SMV case. Different labeled curves correspond to different numbers of sensors N.

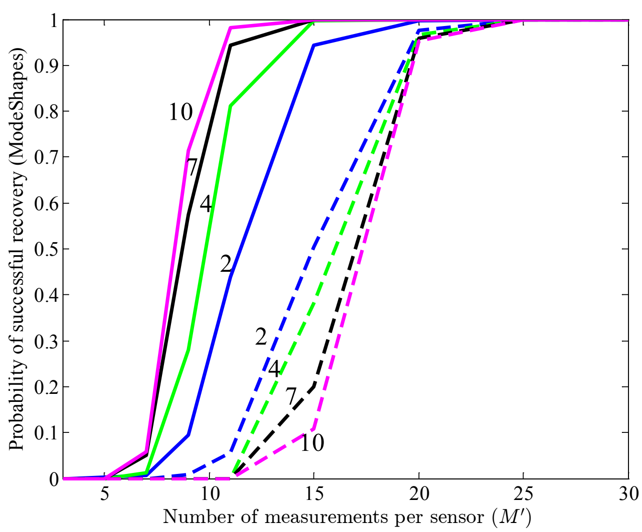

Fig. 2. Random temporal compression: recovery of mode shapes via joint ANM (solid lines) and separate ANM (dashed lines). The problem parameters as the same as in Figure 1.

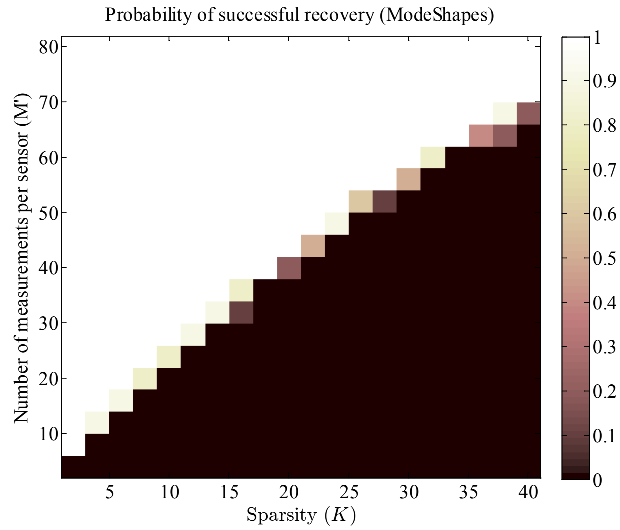

Fig. 3. Random temporal compression: the probability of successful recovery for mode shapes with ANM when $N = 10$ is fixed. We set $M' = 4 : 4 : 80$, $K = 2 : 2 : 20$ and perform 10 trials for each M' and K. For each sparsity level K, we randomly pick K normalized frequencies from a frequency set $\mathcal{F} = 0.03 : \mathrm{MinSep} : 0.99$, where $\mathrm{MinSep} = 2/M$ denotes the minimum separation. The amplitudes $A_k, k = 1, \ldots, K$ are set as K random numbers. $M = 100$ uniform samples are taken with a sampling rate of 100 Hz before compression.

References

[1] H. Sohn, C. R. Farrar, F. M. Hemez, D. D. Shunk, D. W. Stinemates, B. R. Nadler, and J. J. Czarnecki, "A review of structural health monitoring literature: 1996–2001," *Los Alamos National Laboratory, USA*, 2003.

[2] C. R. Farrar and K. Worden, "An introduction to structural health monitoring," *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 365, no. 1851, pp. 303–315, 2007.

[3] J. P. Lynch and K. J. Loh, "A summary review of wireless sensors and sensor networks for structural health monitoring," *Shock and Vibration Digest*, vol. 38, no. 2, pp. 91–130, 2006.

[4] J. Y. Park, M. B. Wakin, and A. C. Gilbert, "Modal analysis with compressive measurements," *IEEE Transactions on Signal Processing*, vol. 62, no. 7, pp. 1655–1670, 2014.

[5] G. Tang, B. N. Bhaskar, P. Shah, and B. Recht, "Compressed sensing off the grid," *IEEE Transactions on Information Theory*, vol. 59, no. 11, pp. 7465–7490, 2013.

[6] R. Heckel and M. Soltanolkotabi, "Generalized line spectral estimation via convex optimization," *arXiv preprint arXiv:1609.08198*, 2016.

[7] Z. Yang and L. Xie, "Exact joint sparse frequency recovery via optimization methods," *arXiv preprint arXiv:1405.6585*, 2014.

[8] Y. Li and Y. Chi, "Off-the-grid line spectrum denoising and estimation with multiple measurement vectors," *IEEE Transactions on Signal Processing*, vol. 64, no. 5, pp. 1257–1269, 2016.

[9] S. Li, D. Yang, G. Tang, and M. B. Wakin, "Atomic norm minimization for modal analysis from random and compressed samples," *arXiv preprint arXiv:1703.00938*, 2017.