Reconstructing Signals from a Union of Linear Subspaces Using a Generalized CoSaMP

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I. Introduction

In this work we consider the problem of recovering an unknown signal $\boldsymbol{x} \in \mathbb{R}^n$ from m linear noisy measurements of the form

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{e}, \tag{1}$$

where $\boldsymbol{A}$ is an $m \times n$ matrix with entries independently drawn from the standard normal distribution and $\boldsymbol{e} \in \mathbb{R}^m$ represents the noise. We assume that the noise does not depend on $\boldsymbol{A}$, and that its energy is bounded. Further, we assume that $m < n$. As in this case recovering $\boldsymbol{x}$ from $\boldsymbol{y}$ becomes an ill-posed problem, an additional prior on $\boldsymbol{x}$ is required. We rely on a general signal model - a union of low dimensional linear subspaces [1]–[3]. More specifically, we assume that a possibly infinite set of finite-dimensional subspaces $\mathcal{S} = \{\mathcal{V}_i\}$ is given, and that the signal belongs to one of the subspaces in $\mathcal{S}$, i.e., $\boldsymbol{x} \in \mathcal{V}_0$, and $\mathcal{V}_0 \in \mathcal{S}$. However, this subspace is unknown. We define the B-order sum for the set $\mathcal{S}$, with an integer $B \geq 1$, as

$$\mathcal{S}^B \triangleq \left\{ \sum_{i=1}^{B} \mathcal{V}_i \, : \, \mathcal{V}_i \in \mathcal{S} \right\}. \tag{2}$$

We also define the union of all subspaces in the set $\mathcal{S}^B$ as

$$\mathcal{U}^B \triangleq \left\{ \bigcup_i \mathcal{V}_i^B \, : \, \mathcal{V}_i^B \in \mathcal{S}^B \right\}. \tag{3}$$

Notice that our signal model is $\boldsymbol{x} \in \mathcal{U}$, where $\mathcal{U} \equiv \mathcal{U}^1$. This general union of subspaces framework subsumes many popular models, such as sparse representation and structured sparsity [4], [5], the low-rank structure [6], the signal space setup [7], the cosparse analysis framework [8], and also combinations of these models.

II. Main Results

We study a generalized version of the popular compressive sampling matching pursuit (CoSaMP) [9]. The generalized CoSaMP (GCoSaMP) algorithm is presented in Table I. The main difference compared to CoSaMP is that support selections are replaced by subspace selections. The notation $\boldsymbol{P}_{\mathcal{V}}$ in GCoSaMP stands for the orthogonal projection onto the subspace $\mathcal{V}$.

Denoting the unit Euclidean sphere in $\mathbb{R}^n$ by $\mathbb{S}^{n-1}$, we provide recovery guarantees that depend on the Gaussian mean width of the set $\mathcal{K} \triangleq \mathcal{U}^4 \cap \mathbb{S}^{n-1}$, defined as

$$w(\mathcal{K}) \triangleq \mathrm{E}_{\boldsymbol{g}} \left\{ \sup_{\boldsymbol{z} \in \mathcal{K}} \langle \boldsymbol{g}, \boldsymbol{z} \rangle \right\}, \tag{4}$$

where the expectation is taken over $\boldsymbol{g} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_n)$ [10], [11]. We show that when the number of measurements m is large enough with respect to some constant $m_0 = \mathcal{O}(w^2(\mathcal{K}))$, we have that the reconstruction result $\hat{\boldsymbol{x}}$ satisfies

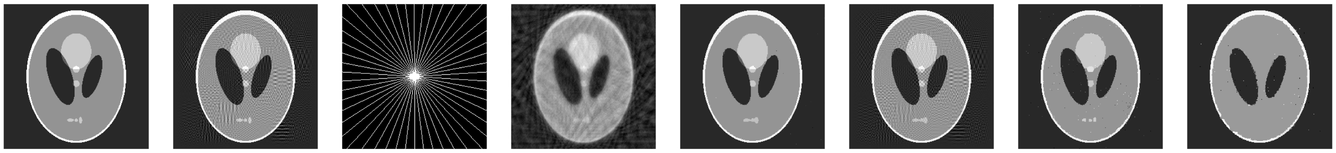
$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_2 \leq c_0 + c_1 \|\boldsymbol{e}\|_2 \tag{5}$$

with high probability, where $c_0 = \mathcal{O}(m^{-t/2})$ tends to zero as the number of iterations t grows, and $c_1 = \mathcal{O}(m^{-1})$. The significance of this result, except for its generality, is in showing that the effect of any stationary noise, that does not depend on $\boldsymbol{A}$, tends to zero as the number of measurements grows. The recovery guarantees hold for any algorithm that is an exact instance of GCoSaMP. This is the case in CoSaMP [9] and ADMiRA [12], though the proof techniques previously used for them did not lead to this general conclusion on the noise reduction except for the special case of Gaussian noise [13].

The computational complexity of GCoSaMP is dominated by the complexity of the subspace selections problem. For certain choices of $\mathcal{S}$, optimal subspace selection methods exist and can be implemented efficiently, while for other choices of $\mathcal{S}$, relaxations of GCoSaMP that rely on approximate subspace selection strategies are required in order to obtain practical algorithms. These relaxed versions may not possess the aforementioned recovery guarantees, but they are still closely related to GCoSaMP. Examples of relaxed algorithms include SSCoSaMP [7] and ACoSaMP [14]. Following this route, we apply GCoSaMP for signal reconstruction in a *combined model*, $\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2$, where $\boldsymbol{x}_1$ is a sparse-synthesis signal (in a given dictionary) and $\boldsymbol{x}_2$ is a cosparse-analysis signal (in a given analysis operator). We use an approximate subspace selection strategy that finds the support and cosupport separately using a simple thresholding for each of them. We name the resulted method synthesis-analysis CoSaMP (SACoSaMP). Though it does not formally possess the theoretical guarantees we derived for GCoSaMP, it is shown to be useful in several scenarios.

III. Experiments

We focus on the application of SACoSaMP to simultaneous image reconstruction and structured noise removal. The noise is assumed to be a random combination with random Gaussian weights of a small number (500) of atoms from the local discrete cosine transform (DCT) with window size of 64×64 pixels and overlap of 32 pixels (the 16×16 lowest frequencies in each window are excluded from the dictionary). The noise is added to an original clean image, and the resulted noisy image is quantized to 8 bits per pixel (bpp). The sampling operator $\boldsymbol{A}$ is a 2D Fourier transform that measures only part of the points in the Fourier domain according to a given binary mask. The cosparse analysis operator is the finite difference analysis operator that computes horizontal and vertical discrete derivatives of an image. We examined the performance of several reconstruction methods for two different test images. The inputs and the results are given in Fig. 1. More details on this work appear in [15].



(a) *Phantom* (b) *Phantom* with textured noise (c) Sampling mask (d) Naïve recovery (e) SACoSaMP - *phantom* recovery (f) SACoSaMP - *phantom* + texture recovery (g) Modified SACoSaMP - *phantom* recovery (h) ACoSaMP recovery

(i) *House* (j) *House* with textured noise (k) Sampling mask (l) Naïve recovery (m) SACoSaMP - *house* recovery (n) SACoSaMP - *house* + texture recovery (o) Modified SACoSaMP - *house* recovery (p) ACoSaMP recovery

Fig. 1: Recovery results for the modified Shepp-Logan phantom (top) and house (bottom) images with SNR of 7dB and 10dB respectively. From left to right: Original image, noisy image (with textured noise), binary mask for Fourier domain sampling, naïve recovery using zero padding and inverse Fourier transform, SACoSaMP recovery of the analysis part (original image), SACoSaMP recovery of the analysis + synthesis parts (noisy image), modified-SACoSaMP (with split least squares step) recovery of the analysis part (original image), and ACoSaMP recovery (analysis part). The algorithms' parameters are tuned manually.

Input: $\boldsymbol{A}, \boldsymbol{y}$, stopping criterion, set of subspaces $\mathcal{S}$, where $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{e}$, such that $\boldsymbol{e}$ is an additive noise and $\boldsymbol{x}$ is an unknown signal satisfying $\boldsymbol{x} \in \mathcal{U}$.

Output: $\hat{\boldsymbol{x}} \in \mathcal{U}$ an estimate for $\boldsymbol{x}$.

Initialize: $\boldsymbol{r} = \boldsymbol{y}, \boldsymbol{x}^0 = 0, t = 0, \mathcal{V}^0 = \varnothing$

while *stopping criterion not met* **do**

$t = t + 1;$

$\tilde{\boldsymbol{v}} = \boldsymbol{A}^*\boldsymbol{r};$

$\mathcal{V}^t_\Delta = \underset{\mathcal{V}\in\mathcal{S}^2}{\operatorname{argmin}} \|\tilde{\boldsymbol{v}} - \boldsymbol{P}_\mathcal{V}\tilde{\boldsymbol{v}}\|_2;$

$\tilde{\mathcal{V}}^t = \mathcal{V}^{t-1} + \mathcal{V}^t_\Delta;$

$\tilde{\boldsymbol{x}}^t = \underset{\boldsymbol{z}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{z}\|_2 \ \ \text{s.t.} \ \ \boldsymbol{z} \in \tilde{\mathcal{V}}^t;$

$\mathcal{V}^t = \underset{\mathcal{V}\in\mathcal{S}}{\operatorname{argmin}} \|\tilde{\boldsymbol{x}}^t - \boldsymbol{P}_\mathcal{V}\tilde{\boldsymbol{x}}^t\|_2;$

$\boldsymbol{x}^t = \boldsymbol{P}_{\mathcal{V}^t}\tilde{\boldsymbol{x}}^t;$

$\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^t;$

end

$\hat{\boldsymbol{x}} = \boldsymbol{x}^t;$

TABLE I: Generalized CoSaMP (GCoSaMP)

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