Compressed Sensing of FRI Signals using Annihilating Filter-based Low-rank Interpolation

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I. Introduction

The spectral compressed sensing (CS) or compressed sensing off-the-grid deals with robust reconstruction of the underlying signal composed of Diracs from sparse Fourier domain measurements [1]–[4]. The stream of Diracs is a special instance of signals with a finite rate of innovation (FRI) [5]–[7]. The class of FRI signals includes a stream of Diracs, a stream of differentiated Diracs, non-uniform splines, and piecewise smooth polynomials. Although Vetterli et al. [5]–[7] proposed *time-domain* sparse sampling schemes, the extension of the spectral compressed sensing for general class of FRI signals is not available. Therefore, one of the main aims of this paper is to generalize the scheme by Vetterli et al. [5]–[7] to address Fourier CS problems that recover a general class of FRI signals from irregularly subsampled Fourier measurements.

II. Main Results

The Fourier CS problem of our interest is to recover the unknown signal $x(t)$ from the Fourier measurement:

$$\hat{x}(f) = \mathcal{F}\{x(t)\} = \int x(t)e^{-i2\pi ft}dt \ .$$

Without loss of generality, we assume that the support of $x(t)$ is $[0,1]$. Then, the sampled Fourier data at the Nyquist rate is defined by $\hat{x}[k] = \hat{x}(f)|_{f=k}$. We also define a length $(r+1)$-annihilating filter $\hat{h}[k]$ for $\hat{x}[k]$ that satisfies

$$(\hat{h} * \hat{x})[k] = \sum_{l=0}^{r} \hat{h}[l]\hat{x}[k-l] = 0, \quad \forall k. \tag{1}$$

The explicit form of the minimum length annihilating filter has been extensively studied for various FRI signals [5]–[7].

Suppose that the filter $\hat{h}[k]$ is the minimum length annihilating filter. Then, for any $k_1 \geq 1$ tap filter $\hat{a}[k]$, it is easy to see the filter $\hat{h}_a = \hat{a} * \hat{h}$ with $d = r + k_1 - 1$ taps is also an annihilating filter for $\hat{x}[k]$, because $\hat{h}_a * \hat{x} = \hat{a} * \hat{h} * \hat{x} = 0$. The corresponding matrix representation provides us $\mathscr{H}(\hat{\mathbf{x}})\bar{\hat{\mathbf{h}}}_a = \mathbf{0}$, where $\hat{\mathbf{h}}_a = \left[\hat{h}_a[0], \cdots, \hat{h}_a[d-1]\right]^T$ and the Hankel structure matrix $\mathscr{H}(\hat{\mathbf{x}})$ is constructed as

$$\mathscr{H}(\hat{\mathbf{x}}) = \begin{bmatrix} \hat{x}[0] & \hat{x}[1] & \cdots & \hat{x}[d-1] \\ \hat{x}[1] & \hat{x}[2] & \cdots & \hat{x}[d] \\ \vdots & \vdots & \ddots & \vdots \\ \hat{x}[n-d] & \hat{x}[n-d+1] & \cdots & \hat{x}[n-1] \end{bmatrix} \tag{2}$$

In [8], we then show the following key result :

Theorem 1. *Let $r+1$ denote the minimum size of annihilating filter that annihilates sampled Fourier data $\hat{x}[k]$. Assume that $\min\{n-d+1, d\} > r$. Then, for a given Hankel structured matrix $\mathscr{H}(\hat{\mathbf{x}})$ constructed in* (2)*, we have*

$$\textsc{rank}\mathscr{H}(\hat{\mathbf{x}}) = r. \tag{3}$$

When the underlying FRI signals has cardinal representations (i.e. singularities are located only on uniform grid), Theorem 1 holds using a wrap-around Hankel matrix. For general class FRI signals, there exist *whitening* operators that convert the FRI signals to sparse innovations such as Diracs or differentiated Diracs [5], [9]. Then, the associated annihilating filters exist in the weighted Fourier domain, where the weighting is determined by spectrum of the associated whitening operator [5], [9]. Therefore, Theorem 1 informs that there always exists a low-rank Hankel structured matrix in *weighted* Fourier domain, and the associated rank is determined by the minimum length annihilating filter in the weighted Fourier domain.

Accordingly, the only required change for CS recovery of FRI signals is an additional Fourier domain interpolation step that estimates missing Fourier measurements. This can be done using the following low-rank Hankel matrix completion algorithm.

$$\begin{array}{ll} \underset{\hat{\mathbf{g}} \in \mathbb{C}^n}{\text{minimize}} & \|\mathscr{H}(\hat{\mathbf{g}})\|_* \\ \text{subject to} & P_\Omega(\hat{\mathbf{g}}) = P_\Omega(\hat{\mathbf{l}} \odot \hat{\mathbf{x}}). \end{array} \tag{4}$$

where $\|\cdot\|_*$ denotes the matrix nuclear norm, $\hat{\mathbf{l}}$ is the spectrum of the whitening operator, and $\odot$ is element-wise product. Once a set of weighted Fourier measurements at consecutive frequencies is interpolated, the element-wise weights are removed and a FRI signal can be reconstructed using Prony's method and matrix pencil algorithms, as in earlier studies [5]–[7].

In [8], we further show that the proposed Fourier CS of FRI signals operates at a near-optimal rate with provable performance guarantees.

Theorem 2. *Let Fourier domain sampling index Ω be a multi-set consisting of random indices following the uniform distribution on $\{0, \ldots, n-1\}$. Suppose, furthermore, that $\mathscr{H}(\hat{\mathbf{l}} \odot \hat{\mathbf{x}})$ is of rank-r and satisfies the standard incoherence condition [10] with parameter μ. Then there exists an absolute constant c_1 such that $\hat{\mathbf{l}} \odot \hat{\mathbf{x}}$ is the unique minimizer to* (4) *with probability $1 - 1/n^2$, provided*

$$m \geq c_1 \mu c_s r \log^\alpha n, \tag{5}$$

where $\alpha = 2$ if Hankel matrix has the wrap-around property; $\alpha = 4$, otherwise, and $c_s := \max\{n/n_1, n/n_2\}$.

III. Conclusion

This paper developed a near-optimal Fourier CS framework using a structured low-rank interpolator in the measurement domain to use before an analytic reconstruction procedure is applied. Numerical results in Appendix confirmed that the proposed method outperforms the existing CS approaches.

APPENDIX

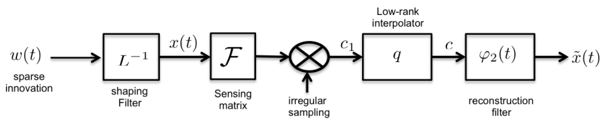

Fig. 1. Proposed sampling scheme: here, the CS step is replaced by a discrete low-rank interpolator, and the final reconstruction is done using the reconstruction filter from fully sampled data.

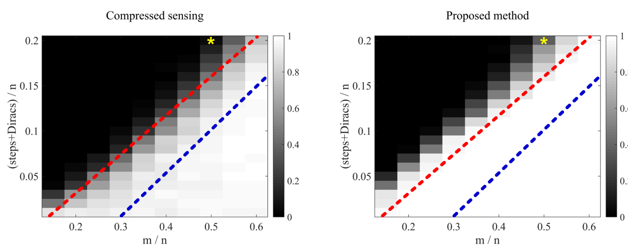

Fig. 2. Phase-transition diagrams for recovering the super-position of the piecewise constant signal and Diracs from m randomly sampled Fourier samples. The size of the target signal (n) is 100 and the annihilating filter size d was set to 51. The left and right graphs correspond to the phase-transition diagram of the l_1-TV compressed-sensing approach and the proposed low-rank interpolation approach, respectively. The success ratio is obtained from the success ratio of 300 Monte Carlo runs. Two transition lines from compressed sensing (blue) and the low-rank interpolator (red) are overlaid.

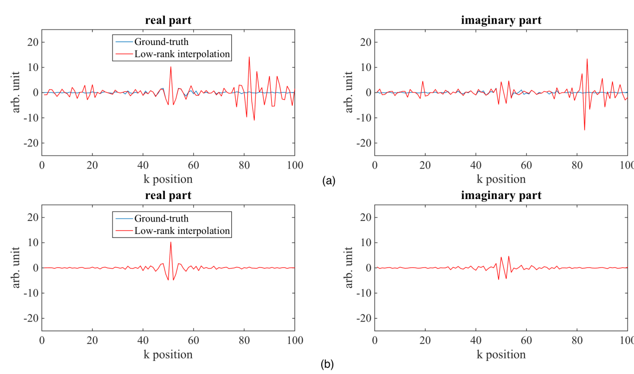

Fig. 3. Fully sampled Fourier measurement and the interpolated data from $m = 36$ irregularly sampled data. (a) Low-rank interpolation results without spectrum weighting. (b) The proposed low-rank interpolation using optimal weighting $\hat{l}(\omega) = i\omega$. For this simulation, the following parameters were used: $d = 51$, $n = 100$ and $m = 36$.

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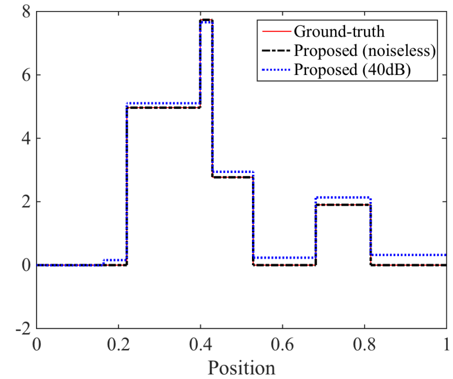

Fig. 4. Proposed reconstruction result of a piecewise constant signal from noiseless and 40dB noisy sparse Fourier samples. For this simulation, the following parameters were used: $d = 51$, $n = 100$ and $m = 36$. The matrix pencil approach was used for signal recovery after the missing Fourier data was interpolated using the proposed low-rank interpolation scheme with optimal weighting $\hat{l}(\omega) = i\omega$.

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