Sampling from binary measurements - On Reconstructions from Walsh coefficients

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***Abstract*—High quality reconstructions from a small amounts of measurements offers interesting possibilities in applications such as medical imaging [18], single-pixel and lensless cameras [11] or fluorescence microscopy [21]. This is possible due to techniques such as generalized sampling [1], [10], [15], [23], and compressed versions of this [4], [11], as well as methods based on data assimilation and additional information modelled by PDEs [5], [6], [9]. All of these methods have in common that the reconstruction quality depends highly on the subspace angle between the sampling and the reconstruction space. Fortunately, one can show that for binary measurements, modelled with Walsh functions and wavelets, the relation between the amount of data sampled and the coefficients reconstructed has to be only linear to ensure that the angle is bounded from below and hence the reconstruction is accurate and stable.**

I. Main Results

During the last decades the connection between sampling and popular reconstruction bases and frames such as wavelets [20], shearlets [16] and curvelets [7] have been emphasised. In particular, in many applications the samples are dictated by the physics of the acquisition device, however, the signal may be best represented in function systems as mentioned above. Thus, the desire to convert sampling information to coefficients of the signal in X-let frames has been a strong motivation to build new sampling and reconstruction techniques. This includes the finite section method [13], [14], [17], generalized sampling [1]–[3], [10], [15], [19], [22], [24] and data assimilation techniques [5], [6], [9]. For the latter two schemes the choice of the amount of data acquired according to the amount of data reconstructed is crucial. This relation is called the stable sampling rate, as it defines the amount of samples needed for a stable and accurate reconstruction. We show that for the reconstruction from binary measurements and wavelets the stable sampling rate is linear. Binary measurements, after a simple subtraction trick, can be converted to a 1 and -1 setup that is modelled by the Hadamard transform. The kernel of the Hadamard transform is given by Walsh functions. These measurements arise in several applications as lensless camera and single-pixel cameras [11] and fluorescence microscopy [21].

In the setting of this paper we deal with the reconstruction of a function $f \in L^2([0,1]^d)$ from linear meausrements $m_i(f) = \langle s_i, f\rangle$, $i \in \mathbb{N}$. The reconstruction space $\mathcal{R}$ is spanned by functions $\{r_i\}_{i\in\mathbb{N}}$ and the sampling space $\mathcal{S}$ is spanned by functions $\{s_i\}_{i\in\mathbb{N}}$. The corresponding spaces of the first N or M elements are denoted by $\mathcal{R}_N$ and $\mathcal{S}_M$. As reconstruction space we use boundary Wavelets [8] due to their great representation properties. To model the binary measurements we utilize Walsh functions [12].

Definition 1 (Walsh function): Let $n = \sum_{i\in\mathbb{Z}} n_i 2^{i-1}$ with $n_i \in \{0,1\}$ be the dyadic expansion of $n \in \mathbb{R}$. Analogously, let $x = \sum_{i\in\mathbb{Z}} x_i 2^{i-1}$ with $x_i \in \{0,1\}$. The generalized Walsh functions in $L^2(\mathbb{R})$ are given by

$$\mathrm{Wal}(n,x) = (-1)^{\sum_{i\in\mathbb{Z}}(n_i+n_{i+1})x_{-i-1}}. \tag{1}$$

We extend it to functions in $L^2([0,1]^d)$ by the tensor product for $n = (n_k)_{k=1,\dots,d}, x = (x_k)_{k=1,\dots,d}$

$$\mathrm{Wal}(n,x) = \bigotimes_{k=1}^{d} \mathrm{Wal}(n_k, x_k). \tag{2}$$

Walsh functions obey a lot of useful properties in the dyadic analysis, while wavelets are continuous in the decimal analysis. Therefore, the two systems do not seem to work well together from a first glance. Nevertheless, we show that the reconstruction from Walsh functions with wavelets works very accurate and stable. Besides a good choice of the sampling and reconstruction space the reconstruction method plays the main role for satisfying results. Two main points to compare reconstruction methods are the stability and accuracy. The first is measured by the condition number κ and the second by the difference to the optimal solution, i.e. the orthogonal projection on the reconstruction space $P_{\mathcal{R}_N}$.

Now, we consider two reconstruction methods, which are both proven to be optimal. Moreover, they share the fact that the error bound depends on the subspace angle $\cos(\omega(\mathcal{R}_N, \mathcal{S}_M)) = \inf_{r\in\mathcal{R}_N, ||r||=1} ||P_{\mathcal{S}_M} r||, \quad \omega(\mathcal{R}_N, \mathcal{S}_M) \in [0, \pi/2]$ between the sampling and the reconstruction space. For the reconstruction with generalized sampling $G_{N,M}(f)$ one has the error estimate

$$||f - G_{N,M}(f)|| \leq \frac{1}{\cos(\omega(\mathcal{R}_N, \mathcal{S}_M))} ||f - P_{\mathcal{R}_N} f|| \tag{3}$$

and the condition number κ is given by $\kappa = 1/\cos(\omega(\mathcal{R}_N, \mathcal{S}_M))$. The reconstruction method A^* published in [6] fulfils

$$||f - A^*(P_{\mathcal{S}_M} f)|| \leq \frac{1}{\cos(\omega(\mathcal{R}_N, \mathcal{S}_M))} \operatorname{dist}(f, \mathcal{R}_N). \tag{4}$$

Hence, it it natural to investigate the relation between N and M such that the subspace angle is bounded from below. This relation is called the stable sampling rate, i.e.

$$\Theta(N,\theta) = \min\left\{ M \in \mathbb{N} : \cos(\omega(\mathcal{R}_N, \mathcal{S}_M)) > \frac{1}{\theta} \right\}. \tag{5}$$

The main theorem shows, that the stable sampling rate is linear for the case of Walsh functions and boundary Wavelets.

Theorem 1: Let $\mathcal{S}$ and $\mathcal{R}$ be the sampling and reconstruction spaces of Walsh functions and boundary Wavelets in $L^2(\mathbb{R}^d)$. If $N = 2^{dR}$ with some $R \in \mathbb{N}$ then for every $\theta \in (1, \infty)$ there exist S_θ such that $\Theta(N;\theta) \leq S_\theta N = \mathcal{O}(N)$, i.e. the stable sampling rate is linear.

With the results in [2], [3] we now have a broad knowledge about the accuracy and stability for two major applications of sampling theory, i.e. systems with Fourier samples and those with binary measurements.

REFERENCES

[1] B. Adcock, A.C. Hansen *A Generalized Sampling Theorem for Stable Reconstructions in Arbitrary Bases* J. Fourier Anal. Appl. 18, (4), 685-716, 2010

[2] B. Adcock, A.C. Hansen, G. Kutyniok, J. Ma *Linear Stable Sampling Rate: Optimality of 2D Wavelet Reconstructions from Fourier Measurements* SIAM J. Math. Anal. 47, (2), 1196-1233, 2015

[3] B. Adcock, A.C. Hansen, C. Poon *On optimal wavelet reconstructions from Fourier samples: linearity and universality* Appl. Comput. Harmon. Anal. 36, (3), 387-415, 2014

[4] B. Adcock, A. C. Hansen, C. Poon, B. Roman. Breaking the coherence barrier: A new theory for compressed sensing. *Forum of Mathematics, Sigma*, (to appear)

[5] M. Bachmayr, A. Cohen, R. deVore, G. Migliorati *Sparse polynomial approximation of parametric elliptic PDEs. Part II: lognormal coefficients* arXic:1509.07050, 2015

[6] P. Binev, A. Cohen, W. Dahmen, R. deVore, G. Petrova, P. Wojtaszczyk *Data Assimilation in Reduced Modeling* SIAM UQ, to appear, 2015

[7] E. J. Candès, L. Demanet *Curvelets and Fourier Integral Operators* C. R. Acad. Sci., 336, (1), 395-398, 2003

[8] A. Cohen, I. Daubechies, P. Vial. *Wavelets on the Interval and Fast Wavelet Transforms* Appl. Comput. Harmon. Anal. 1, (1), 54 - 81, 1993

[9] R. deVore, G. Petrova, P. Wojtaszczyk *Data assimilation and sampling in Banach spaces* arXiv:1602.06342, 2016

[10] Y. Eldar. Sampling with arbitrary sampling and reconstruction spaces and oblique dual frame vectors. *J. Fourier Anal. Appl.*. 9(1): 77–96, 2003

[11] S. Foucart, H. Rauhut *A Mathematical Introduction to Compressive Sensing* New York: Springer Science+Business Media, 2013

[12] E. Gauss *Walsh Funktionen für Ingenieure und Naturwissenschaftler* Wiesbaden: Springer Fachmedien, 1994

[13] R. Hagen, S. Roch, and B. Silbermann *C^*-algebras and numerical analysis* Monogr. Textbooks Pure Appl. Math. 236, Dekker, New York, 2001

[14] A. C. Hansen *On the approximation of spectra of linear operators on Hilbert spaces* J. Funct. Anal. 254, (8), 2092 - 2126, 2008

[15] T. Hrycak, K. Gröchenig. Pseudospectral Fourier reconstruction with the modified Inverse Polynomial Reconstruction Method. *J. Comput. Phys.*, 229(3): 933–946, 2010

[16] G. Kutyniok, W.-Q Lim *Compactly Supported Shearlets are Optimally Sparse* J. Approx. Theory 163, 1564 - 1589, 2011

[17] M. Lindner *Infinite matrices and their finite sections: An introduction to the limit operator method* Basel: Birkh äuser Verlag, 2006

[18] M. Lustig, D. Donoho, J. M. Pauly. *Sparse MRI: the application of compressed sensing for rapid MR imaging* Magnetic Resonance in Medicine, 58, (6), 11821195, 2007

[19] J. Ma. Generalized sampling reconstruction from Fourier measurements using compactly supported shearlets. *Appl. Comput. Harmon. Anal.* (to appear)

[20] S. Mallat *A wavelet tour of signal processing* San Diego: Academic Press, 1998

[21] M. Müller *Introduction to Confocal Fluorescence Microscopy* Bellingham, Washington: SPIE, 2006

[22] C. Poon. A consistent, and stable approach to generalized sampling. *J. Fourier Anal. Appl.* 20(5), 9851019, 2014

[23] C.E. Shannon *A mathematical theory of communication* Bell Syst. Tech. J. 27, 379 - 423, 623 - 656, 1948

[24] M. Unser, A. Aldroubi. A general sampling theory for nonideal acquisition devices. *IEEE Trans. Signal Process.* 42(11):29152925, 1994.