An Intrinsic Model for the Infimal Convolution of First and Second Order Differences on Manifold-Valued Images

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***Abstract*—Infimal convolution type functions were successfully applied in regularization terms of variational models for restoring and decomposing images. In this paper, we generalize the infimal convolution of first and second order differences to manifold-valued images in a intrinsic way, i.e. without embedding the manifold into an Euclidean space. We apply a gradient descent algorithm to find a critical point of the corresponding functional and demonstrate the approach by numerical examples on the 2-sphere and the manifold of positive definite matrices with the affine invariant metric.**

I. Introduction

Due to the large amount of applications variational methods have gained a lot of interest in image processing in recent years. Typically these models consist of a data fitting term and a regularization term also known as prior. In this paper we restrict our attention to least squares data fitting terms. Starting with methods having first order derivatives in their regularization term like the total variation (TV), higher order derivatives were incorporated into the approach to cope with the staircasing effect caused by TV regularization and to better adapt to specific applications. Besides additive coupling of higher order derivatives, their infimal convolution (IC) [5] or their total generalized variation (TGV) [4], [8] were proposed in the literature.

With the emerging possibilities to capture different modalities of data, image processing methods are transferred to the case where the measurements (pixels) take values on Riemannian manifolds. Examples are Interferometric Synthetic Aperture Radar (InSAR) with values on the circle S^1, directional data on the 2-sphere S^2, electron backscatter diffraction (EBSD) with data on quotient manifolds of $\mathrm{SO}(3)$ or diffusion tensor magnetic resonance imaging (DT-MRI), where each measurement is a symmetric positive definite 3×3 matrix.

In this paper we modify the IC of TV-like terms with first and second order differences for manifold-valued images. A straightforward idea is to embed the d-dimensional manifold into $\mathbb{R}^n$, $n > d$, so that the Euclidean arithmetic can be applied, and to add a constraint that the resulting image values have to lie on the manifold. Such a model with first order differences was proposed, e.g. in [7]. A drawback of this so-called extrinsic method is that the IC components live in the higher dimensional embedding space and not on the manifold. This makes their interpretation and visualization difficult. Therefore we propose an intrinsic model where all actors remain on the manifold. The main question is how to replace the addition in the IC with a suitable operation on manifolds. Similarly as in [2] we make use of the midpoints of geodesics. We apply a gradient descent algorithm to solve the resulting minimization problem which details can be found in [3].

II. Model

Let $\mathcal{G} := \{1,\ldots,N_1\} \times \{1,\ldots,N_2\}$ denote the pixel grid of an image of size $N_1 \times N_2$ and let $N := N_1N_2$. A model for denoising a gray-value image $f : \mathcal{G} \to \mathbb{R}$ typically corrupted by white Gaussian noise is given by

$$E_{\mathrm{IC}}(u) := \frac{1}{2}\|f-u\|_2^2 + \alpha\left(\beta\mathrm{TV}\#(1-\beta)\mathrm{TV}_2\right)(u), \qquad (1)$$

where $\alpha > 0$ and $\beta \in [0,1]$. A minimizer of $E_{\mathrm{IC}}(u)$ is considered as the denoised image. The term

$$(F_1\#F_2)(u) = \inf_{u=v+w}\{F_1(v)+F_2(w)\}.$$

is called infimal convolution (IC) of F_i, $i = 1,2$. In (1) we deal with the IC of

$$\mathrm{TV}(v) := \sum_{p\in\mathcal{G}}\sqrt{\sum_{q\in\mathcal{N}_p}(v_q-v_p)^2}, \qquad (2)$$

where $\mathcal{N}_p := \{p+(0,1), p+(1,0)\} \cap \mathcal{G}$ and

$$\mathrm{TV}_2(w) := \sum_{p\in\mathcal{G}}\sqrt{d_{2,\mathrm{h}}^2(w_p)+d_{2,\mathrm{v}}^2(w_p)}, \qquad (3)$$

where

$$d_{2,\mathrm{h}}(w_p) := \begin{cases} |w_{p-(1,0)} - 2w_p + w_{p+(1,0)}|, & p_1 \in \{2,\ldots,N_1-1\},\\ 0, & \text{else,}\end{cases} \qquad (4)$$

and similarly $d_{2,\mathrm{v}}(w_p)$ in vertical direction by replacing $(1,0)$ by $(0,1)$ and p_1, N_1 by p_2, N_2. This type of IC was first applied in image processing by Chambolle and Lions [5]. In various applications, the individual IC components v and w are of interest, e.g. in motion separation [6] or crack detection [1]. The IC of the TV functional with other functionals, which are often concatenations of norms and certain linear operators, was used also for texture-structure or line-point separation.

In this paper we make a proposal how to modify the IC of TV-like terms with first and second order differences to manifold-valued images. images $f : \mathcal{G} \to \mathcal{M}$ taking values on a connected, complete d-dimensional Riemannian manifold manifold $\mathcal{M}$. Let $\mathrm{dist} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0}$ denote the geodesic distance on $\mathcal{M}$. Replacing the absolute differences in the TV term (2) by geodesic distance $\mathrm{dist} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0}$ on $\mathcal{M}$ we get

$$\mathrm{TV}(v) := \sum_{p\in\mathcal{G}}\sqrt{\sum_{q\in\mathcal{N}_p}\mathrm{dist}^2(v_q,v_p)}.$$

Observing that in the Euclidean case the second order difference of $x,y,z \in \mathbb{R}^d$ can be rewritten as $x - 2y + z = 2(\frac{1}{2}(x+z) - y)$, we define a counterpart for $x,y,z \in \mathcal{M}$ as follows: let $\mathcal{C}_{x,z}$ be the set of mid points of all geodesics joining x and z. Note that the geodesic γ is unique on manifolds with non positive curvature, i.e. Hadamard manifolds. Then we define

$$\mathrm{d}_2(x,y,z) := \min_{c\in\mathcal{C}_{x,z}}\mathrm{dist}(c,y).$$

Using this definition as before for the rows and columns of u, we get

$$\mathrm{d}_{2,\mathrm{h}}(w_p) := \begin{cases} \mathrm{d}_2(w_{p+(1,0)}, w_p, w_{p-(1,0)}), & p_1 \in \{2,\dots,N_1-1\} \\ 0, & \text{else,} \end{cases}$$

and similarly for the vertical differences. We set

$$\mathrm{TV}_2(w) := \sum_{p\in\mathcal{G}} \sqrt{\mathrm{d}_{2,\mathrm{h}}^2(w_p) + \mathrm{d}_{2,\mathrm{v}}^2(w_p)}.$$

Now we may consider the "midpoint infimal convolution" of F_i, $i = 1, 2$, given by

$$F_1 \#_m F_2(u) := \inf_{u\in\mathcal{C}_{v,w}} \{F_1(v) + F_2(w)\},$$

and using $F_1 := \beta\mathrm{TV}$ and $F_2 := (1-\beta)\mathrm{TV}_2$ we define the following functional on the product manifold $\mathcal{M}^N$:

$$\mathcal{E}_{\mathrm{IC}}(u) := \frac{1}{2}\mathrm{dist}^2(u,f) + \alpha\left(\beta\mathrm{TV}\#_m(1-\beta)\mathrm{TV}_2\right)(u).$$

In order to apply a gradient descent algorithm, we smooth these terms as follows:

$$\begin{aligned} \mathrm{TV}_\varepsilon(v) &:= \sum_{p\in\mathcal{G}} \sqrt{\sum_{q\in\mathcal{N}_p} \mathrm{dist}^2(v_p,v_q) + \varepsilon^2}, \\ \mathrm{TV}_{2,\varepsilon}(w) &:= \sum_{p\in\mathcal{G}} \sqrt{\mathrm{d}_{2,\mathrm{h}}^2(w_p) + \mathrm{d}_{2,\mathrm{v}}^2(w_p) + \varepsilon^2}, \quad \varepsilon > 0. \end{aligned}$$

III. Numerical Results

Next we provide two numerical examples. More examples can be found in [3].

We start with a noise-free $\mathbb{S}^2$-valued signal to illustrate the decomposition. We take a signal along three geodesics being great arcs from the north pole to the equator, along the equator, and further to the south pole. The segments are shortened to $\frac{1}{5}$, $\frac{3}{20}$, and $\frac{1}{5}$ of a circle, respectively. Hence we obtain three geodesic segments with jumps in between, see Fig. 1 (top). We apply the IC model with $\alpha = \frac{11}{100}, \beta = \frac{1}{11}$. The result u in Fig. 1 (bottom) approximates f and its decomposition into v and w yields signals that are nearly piecewise constant and piecewise geodesic, respectively.

Next we show results for denoising a signal with values on the manifold of positive definite 2×2 matrices with the affine invariant Riemannian metric. We created a signal u_0 as the midpoint of a signal with four constant parts and one with two geodesic parts. We apply the the IC model to a noisy version f of this signal. The results shown in Fig. 2 are with parameters optimized with respect to the mean squared error $\epsilon := \frac{1}{|\mathcal{G}|}\sum_{p\in\mathcal{G}} \mathrm{dist}^2(u_p, u_{0,p})$.

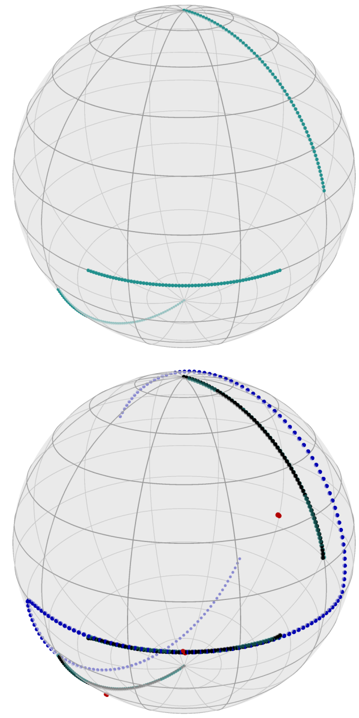

Fig. 1. Decomposition of a signal f of 192 samples (top) on S^1 into two parts: a part v (red) consisting of three nearly constant parts (small TV value) and a nearly piecewise geodesic curve (small TV_2 value) part w (blue). The mid point signal u (black) nearly reconstructs the original signal f (green).

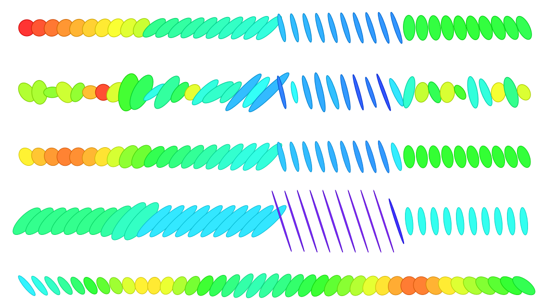

Fig. 2. Denoising of a SPD(2) valued signal f: first row: original signal u_0, second row: noisy signal f corrupted by additive white Gaussian noise. third row: denoised signal $u = c(v,w)$, forth row: piecewise constant part v, and fifth row: geodesic part w.

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