Finite-Valued Sparse Signals

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I. Introduction

Compressed sensing was introduced as an effective method to reconstruct sparse or nearly sparse signals from an underdetermined system of linear equations. In many applications, however, we can assume a second structural constraint besides sparsity, namely that the nonzero entries of x come from a finite or discrete alphabet $\mathcal{A}$. Those signals appear, for example, in error correcting codes [3] as well as massive Multiple-Input Multiple-Output (MIMO) channel [5] and wideband spectrum sensing [2]. A particular example is given by wireless communications, where the transmitted signals are sequences of bits, i.e., with entries in $\mathcal{A} = \{0, 1\}$. However, there also exist several examples of applications, where the transmitted data originate from a finite set $\mathcal{A} \subset \mathbb{R}$ such as in source decoding or radar.

We will focus on the recovery of signals with entries from a bounded lattice using basis pursuit. As it can be proven that post-projecting the solution of basis pursuit does not help to improve performance guarantees [4], we will consider the following adaptation of basis pursuit to recover a signal x with entries in a finite alphabet $\mathcal{A}$

$$\min \|x\|_1 \text{ subject to } Ax = b \text{ and } x \in \operatorname{conv} \mathcal{A}^N. \qquad (P_{\mathcal{A}})$$

We will show that basis pursuit with box constraint can provide highly improved nonuniform recovery guarantees for finite-valued sparse signals. Moreover, it can be shown to be stable under noisy measurements with precise error bounds. Our analysis surprisingly shows that the nonnegative case is very different from the bipolar one. One of our findings is that the positioning of the zero - i.e., whether it is a boundary element or not - is crucial

II. Bipolar Finite Valued Signals

We will first analyze recovery guarantees for *bipolar finite-valued* signals having entries in a general finite alphabet of the form $\mathcal{A} = \{-L_1, \ldots, L_2\} \subset \mathbb{Z}$, with $L_1, L_2 \in \mathbb{N}$. The straightforward adaptation of basis pursuit to bipolar finite-valued signals is given by

$$\min \|x\|_1 \text{ subject to } Ax = b \text{ and } x \in [-L_1, L_2]^N. \qquad (P_{\mathcal{F}})$$

The following variant of the NSP characterizes the solvability of this program and gives an equivalent condition. We will denote $K_j = \{i : x_i = j\}$, $j = -L_1, \ldots, L_2$, and $K = \cup_{i=-L_1}^{L_2} K_i \setminus K_0$.

Definition II.1. *Let $K_{-L_1} \subset K_- \subset [N]$ and $K_{L_2} \subset K_+ \subset [N]$ with $K_- \cap K_+ = \emptyset$. Further let $K = K_- \cup K_+$. A matrix $A \in \mathbb{R}^{m\times N}$ is said to satisfy the* finite NSP *with respect to K_{-L_1}, K_{L_2}, K_- and K_+, if*

$$\ker(A) \cap N_{K_-,K_+} \cap H_{K_{L_2},K_{-L_1}} = \{0\}, \qquad (\mathcal{F}\text{-NSP})$$

where

$$N_{K_-,K_+} = \{w \in \mathbb{R}^N : -\sum_{i\in K_+} w_i + \sum_{i\in K_-} w_i \geq \|w_{K^C}\|_1\},$$

$$H_{K_{L_2},K_{-L_1}} = \{w \in \mathbb{R}^N : w_i \leq 0, i \in K_{L_2}, \textit{and } w_i \geq 0, i \in K_{-L_1}\}.$$

We can utilize this characterization to deduce a sufficient number of measurements for $(P_{\mathcal{F}})$ to succeed in the case where $A \in \mathbb{R}^{m\times N}$ is a Gaussian matrix, by computing the statistical dimension of $N_K \cap H_{K_{L_2},K_{-L_1}}$ and results that stem from [1]. Figure 1 illustrates the phase transition which is determined by the following result.

Theorem II.2. *Let $\varepsilon > 0$, $A \in \mathbb{R}^{m\times N}$ a Gaussian, $b = Ax_0$ and x_0 a bipolar finite-valued signal. Further set $\hat{K} = K \setminus (K_{-L_1} \cup K_{L_2})$, $\hat{k} = |\hat{K}|$ and $k_i = |K_i|$, for $i \in \{-L_1, 0, L_2\}$. $(P_{\mathcal{F}})$ will succeed to recover x uniquely with probability larger than $1-\varepsilon$ if*

$$m \geq \inf_{\tau\geq 0} \left\{ \hat{k}(1+\tau^2) + k_{-L_1} \int_{-\tau}^{\infty} (u-\tau)^2 \phi(u) du + k_{L_2} \int_{-\infty}^{\tau} (u-\tau)^2 \phi(u) du + k_0 \int_{\tau}^{\infty} (u-\tau)^2 \phi(u) du \right\} + \sqrt{8 \log(4/\varepsilon) N}.$$

Note, that signals with entries in $\mathcal{A} = \{-1, 0, 1\}$, which appear often in applications, are a particular instance of bipolar finite-valued signals. For them it holds that $\hat{K} = \emptyset$. Thus, the phase transition for such signals corresponds to the lowest curve $(P_{\pm\text{ter}})$ in Figure 1.

III. Unipolar Finite-Valued Signals

In many applications such as wireless communications or sensor networks we are dealing with even more structured alphabets, namely non-negative alphabets $\mathcal{A} = \{0, \ldots, L\}$, $L \in \mathbb{Z}$, and in particular with unipolar binary $\mathcal{A} = \{0, 1\}$ alphabets. The recovery situation for such signals is quite different from the bipolar one due to the positioning of the zero. We will show that the success of basis pursuit with box constraints for signals with entries in a nonnegative alphabet is equivalent to a weaker NSP condition which subsequently shows that less number of measurements are sufficient. An illustration of the phase transition can be seen in Figure 2. This results contains in particular the case that x is unipolar binary, which was first studied by Stojnic [6]. The phase transition for unipolar binary signals satisfies the lowest curve (P_{bin}) in Figure 2, which illustrates that, independently of the number of non-zeros, $\lceil N/2 \rceil$ measurements are with high probability sufficient to recover a binary signal.

IV. Outlook

Besides introducing equivalent conditions to recover finite-valued sparse signals using basis pursuit with box constraints and computing phase transitions, we will show robustness of the algorithm and we will discuss the importance of the right choice of the exact bounds in the additional constraint. Moreover, we will give some numerical experiments that validate our theoretical results.

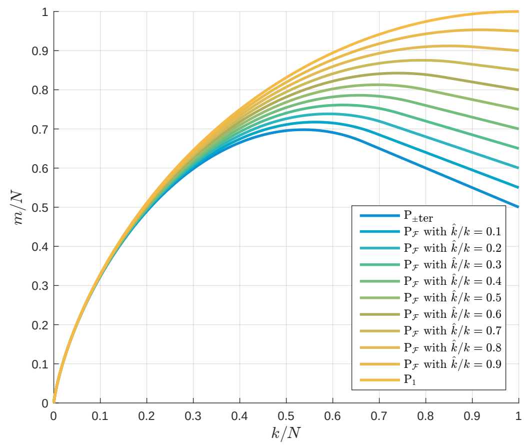

Fig. 1: Phase transition of the convex program $(P_{\mathcal{F}})$ according to the ratio of $\hat{k}$ to k, where k is the size of the whole support of a bipolar finite-valued signal and $\hat{k}$ the number of entries in the signal not equal to zero, to the smallest or to the largest value of the given alphabet. Recovery is likely above the curves.

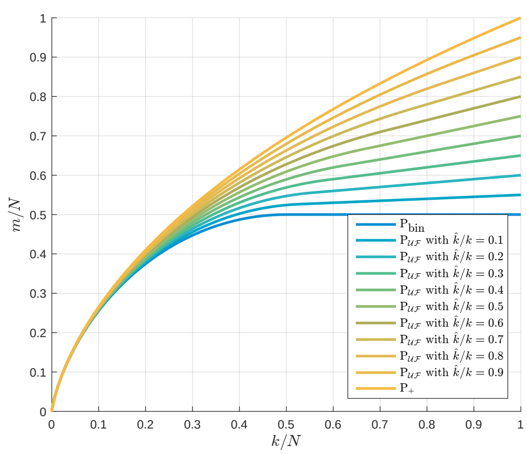

Fig. 2: Phase transition of basis pursuit with box constraints of the from $[0, L]$ according to the ratio of $\hat{k}$ to k, where k is the size of the entire support of a unipolar finite-valued signal and $\hat{k}$ the number of entries in the signal not equal to zero or to the largest value of the given alphabet. Successful recovery related to the area above the curves.

References

[1] D. Amelunxen, M. Lotz, M.B. McCoy and J.A. Tropp, *Living on the edge: Phase transitions in convex programs with random data*, Inf. Inference,Oxford University Press, 2014.

[2] E. Axell, G. Leus, E.G. Larsson, and H.V. Poor, *Spectrum sensing for cognitive radio: State-of-the-art and recent advances*, IEEE Signal Process. Mag., IEEE, vol. 29, 2012.

[3] E.J. Candes, M. Rudelson, T. Tao, R. Vershynin, *Error correction via linear programming*, 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS'05), IEEE, 2005.

[4] A. Flinth and G. Kutyniok, *PROMP: A Sparse Recovery Approach to Lattice-Valued Signals*, 2016.

[5] M. Rossi, A. Haimovich, and Y. Eldar, *Spatial compressive sensing for MIMO radar*, IEEE Trans. Signal Process., vol. 62, IEEE, 2013.

[6] M. Stojnic, *Recovery thresholds for ℓ_1 optimization in binary compressed sensing*, IEEE International Symposium on Information Theory (ISIT), IEEE, 2010.

[7] M. Stojnic, *A simple performance analysis of ℓ_1 optimization in compressed sensing*, IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP), IEEE, 2009.