Stabilizing Embedology: Geometry-Preserving Delay-Coordinate Maps

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Modern science is ingrained with the premise that repeated observations of a dynamic phenomenon can help us understand its underlying mechanisms and predict its future behavior. The field of *nonlinear time-series analysis* models time-series data as observations of the state of a (possibly high-dimensional) deterministic nonlinear dynamical system [1]. It is often postulated that the underlying dynamical system is governed by an attractor that is a low-dimensional geometric subset of the state space, making it reasonable to postulate that temporal dependencies in time-series observations can provide some insight into the structure of the hidden dynamical system. This leads to a fundamental question: *How much information about a hidden dynamical system is available in time-series measurements of the system state?*

Our main contribution is to present a new theoretical result that, for the first time, provides insight into the conditions for when time-series data can (and cannot) be used to reconstruct a geometry-preserving image of the attractor. This result provides formal foundations for many algorithms currently in use for tasks such as time series prediction, as well as giving guidance to choosing algorithmic parameters that are normally chosen heuristically. This is a general result potentially having impact across a wide variety of disciplines, as well providing a concrete bridge between the modern tools used by the compressed sensing community and the classic questions of physics and nonlinear dynamical systems.

Specifically, consider $x(\cdot)$ as the trajectory of a dynamical system in the state space $\mathbb{R}^N$ such that $x(t) \in \mathbb{R}^N$ for $t \in [0, \infty)$. While the system has continuous underlying dynamics, we observe this system at a regular sampling interval $T > 0$. We assume that during the times of interest the state trajectory is contained within a low-dimensional *attractor* [1] $\mathbb{A}$ such that $x(t) \in \mathbb{A} \subset \mathbb{R}^N$ for $t \geq 0$. The attractor $\mathbb{A}$ is assumed to be a bounded, boundary-less, and smooth submanifold of $\mathbb{R}^N$ with $\dim(\mathbb{A}) < N$. In applications of interest we often cannot directly observe this system state but rather receive indirect measurements via a scalar measurement function $h : \mathbb{A} \to \mathbb{R}$. This function generates a single scalar measurement at a regular sampling interval $T > 0$, producing the discrete time series $\{s_i\}_{i \in \mathbb{N}} = \{h(x(i \cdot T))\}_i$, where each $s_i \in \mathbb{R}$. The goal is to "reconstruct" the hidden state trajectory $x(\cdot)$ given only $\{s_i\}_i$. To approach this task, consider the *delay-coordinate map* $F_{h,T,M} : \mathbb{A} \to \mathbb{R}^M$, defined for an integer number of delays M through the relation

$$F_{h,T,M}\left(x(i \cdot T)\right) = \begin{bmatrix} s_i & s_{i-1} & \cdots & s_{i-M+1} \end{bmatrix}^T. \tag{1}$$

Note that the delay-coordinate map is simply formed at a given time by stacking the last M observed time-series values into a vector. Commonly, $\mathbb{R}^M$ is referred to as the *reconstruction space*.

The seminal Takens' embedding theorem [2], [3] asserts that (under very general conditions) it is indeed possible to reconstruct the state space from the time-series data. With this setup, Takens' result roughly states that if $M > 2 \cdot \dim(\mathbb{A})$, then the delay-coordinate map $F_{h,T,M}(\cdot)$ resulting from almost every smooth measurement function $h(\cdot)$ embeds the attractor $\mathbb{A}$ into the reconstruction space $\mathbb{R}^M$ (i.e., the delay-coordinate map forms a diffeomorphism for $\mathbb{A}$). Figure 1 illustrates the concept of a delay-coordinate map in the case of the widely-known Lorenz attractor. While the trajectory in the reconstruction space $F_{h,T,M}(x(\cdot))$ is (topologically) equivalent to the trajectory in the state space $x(\cdot)$—because no two points from the attractor map onto each other in the reconstruction—the mapping could be unstable in the sense that close points may map to points that are far away (and vice versa).

Following the recent literature on various forms of randomized dimensionality reduction [4], [5], we seek conditions under which the delay-coordinate map $F_{h,T,M}(\cdot)$ is a *stable embedding* of the attractor $\mathbb{A}$ by acting as a near-isometry on $\mathbb{A}$:

$$\epsilon_l \leq \frac{\|F_{h,T,M}(x) - F_{h,T,M}(y)\|_2^2}{M \cdot \|x-y\|_2^2} \leq \epsilon_u, \qquad \forall x, y \in \mathbb{A},\ x \neq y \tag{2}$$

for some *isometry constants* $0 < \epsilon_l \leq \epsilon_u < \infty$. If $\epsilon_l \approx \epsilon_u$, the stable embedding condition of (2) guarantees that the delay-coordinate map preserves the *geometry* of the attractor (rather than merely its topology) by ensuring that pairwise distances between points on the attractor are approximately preserved in the reconstruction space.

Our main result gives the conditions on the attractor $\mathbb{A}$, measurement function $h(\cdot)$, number of delays M, and sampling interval T such that $F_{h,T,M}(\cdot)$ is a stable embedding of $\mathbb{A}$. This is a more ambitious objective than Takens' embedding theorem (leading naturally to more restrictive conditions), but with the benefit of quantifying the quality of the embedding and relating that quality to the problem-specific parameters. Roughly speaking (due to space constraints—see [6] for full detail), our main result shows that $F_{h,T,M}(\cdot)$ stably embeds $\mathbb{A}$ (in the sense of (2)) for most measurement functions h, provided that the following condition is satisfied:

$$\mathrm{R}_{H,T,M}(\mathbb{A}) \gtrsim \dim(\mathbb{A}) \cdot \log\left(\frac{\mathrm{vol}(\mathbb{A})^{\frac{1}{\dim(\mathbb{A})}}}{\mathrm{rch}(\mathbb{A})}\right). \tag{3}$$

Here, $\dim(\mathbb{A})$ and $\mathrm{vol}(\mathbb{A})$ are the dimension and volume of the attractor $\mathbb{A} \subset \mathbb{R}^N$, and $\mathrm{rch}(\mathbb{A})$ is an attribute of $\mathbb{A}$ that captures its geometric regularity. To quantify the notion of "most" measurement functions, our result is probabilistic and holds with high probability over measurement functions drawn from a rich probability model H. The *stable rank* $\mathrm{R}_{H,T,M}(\mathbb{A})$ of $\mathbb{A}$ quantifies the ability of the random measurement functions to observe the system attractor. Typically, if a dynamical system is fairly "predictable", then $\mathrm{R}_{H,T,M}(\mathbb{A})$ grows proportionally with M as the number of delays grows. In this case, the delay-coordinate map stably embeds $\mathbb{A}$ when the number of delays scales linearly with the dimension of the attractor as in Takens' original theorem; an example of such behavior is given in Figure 2. On the other hand, if the dynamical system is highly unpredictable, then it is likely that $\mathrm{R}_{H,T,M}(\mathbb{A})$ plateaus rapidly with increasing M and it will be more difficult to stably embed this system through delay-coordinate mapping even with very long delay vectors. As we will discuss, these conditions also have a natural interpretation in the context of classical empirical methods for choosing T and M [7]–[9].

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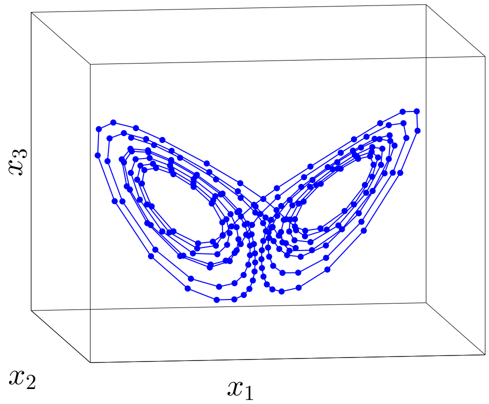
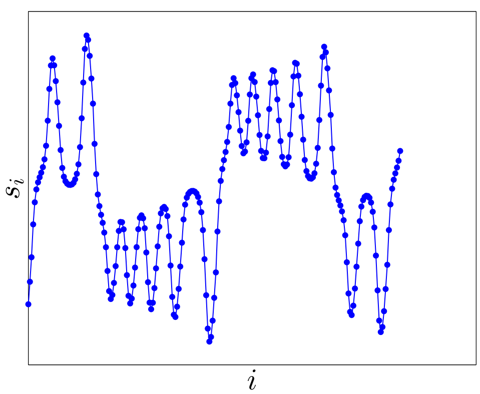
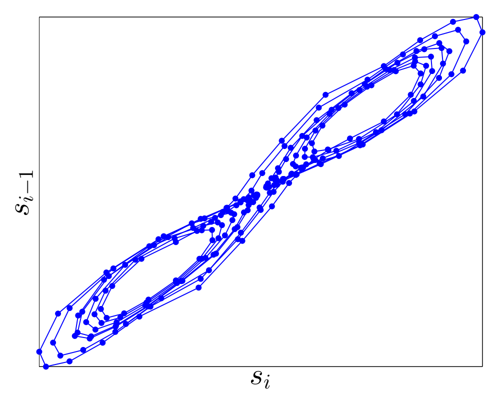


(a) State space

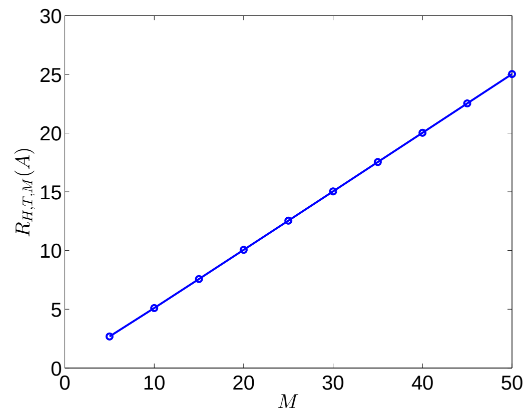


(b) Time series

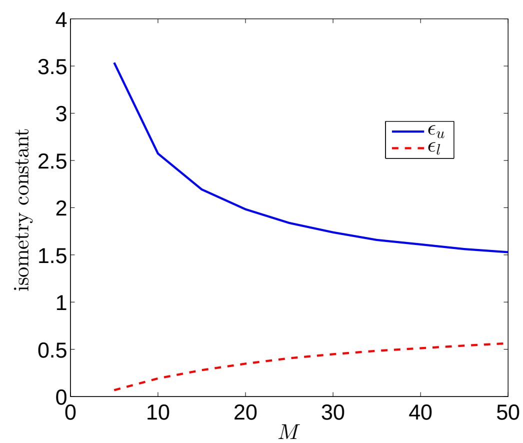


(c) Reconstruction space

Figure 1: *(a) The state space trajectory of the Lorenz attractor in $\mathbb{R}^3$, demonstrating the characteristic butterfly pattern. (b) The time series obtained by a measurement function that only keeps the x_1-coordinate of the trajectory. (c) The delay-coordinate map points with $M = 2$, recreating the butterfly pattern using only the time series.*



(a)



(b)

Figure 2: *Stable rank and quality of delay-coordinate mapping for an example system described in [6]. (a) Stable rank versus M (number of delays) with fixed sampling interval. Note that the stable rank of the system gradually improves with increasing M. (b) Quality of embedding through delay-coordinate mapping as measured by the isometry constants $\epsilon_l \leq \epsilon_u$ versus M (see (2)). Note that, like the stable rank, the quality of embedding gradually improves with increasing M.*

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