PhaseMax: Convex Phase Retrieval Without Lifting

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I. Introduction

Phase retrieval deals with the recovery of an n-dimensional signal $\mathbf{x}^0 \in \mathcal{H}^n$, with $\mathcal{H}$ either $\mathbb{R}$ or $\mathbb{C}$, from $m \geq n$ magnitude measurements of the form [1]

$$b_i = |\langle \mathbf{a}_i, \mathbf{x}^0 \rangle|, \quad i = 1, 2, \ldots, m, \tag{1}$$

where $\mathbf{a}_i \in \mathcal{H}^n$, and $i = 1, 2, \ldots, m$ are measurement vectors.

While the recovery of $\mathbf{x}^0$ from (1) is non-convex, it can be convexified via "lifting" methods that convert the phase retrieval problem to a semidefinite program (SDP). The resulting formulation is convex, but this convexity comes at a high cost: lifting methods square the dimensionality of the problem. As a result, many practical phase retrieval algorithms sacrifice convexity and operate in the original feature space using non-convex methods [2]–[6].

We propose PhaseMax, a formulation of the phase retrieval problem that avoids lifting [7], [8]. Suppose we have some intelligent "guess" $\hat{\mathbf{x}} \in \mathcal{H}^n$ of the solution to (1). We recover the signal $\mathbf{x}^0$ by solving the following convex optimization problem:

$$\text{(PhaseMax)} \quad \begin{cases} \underset{\mathbf{x} \in \mathcal{H}^n}{\text{maximize}} & \langle \mathbf{x}, \hat{\mathbf{x}} \rangle_{\Re} \\ \text{subject to} & |\langle \mathbf{a}_i, \mathbf{x} \rangle| \leq b_i, \quad i = 1, 2, \ldots, m. \end{cases}$$

Here, $\langle \mathbf{x}, \hat{\mathbf{x}} \rangle_{\Re}$ denotes the real-part of the inner product between the vectors $\mathbf{x}$ and $\hat{\mathbf{x}}$. The main idea behind PhaseMax is to find the vector $\mathbf{x}$ that is most aligned with the approximation vector $\hat{\mathbf{x}}$ and satisfies a convex relaxation of the measurement constraints in (1).

Surprisingly, PhaseMax provably recovers the true solution to (1) with high probability. We have the following theorem:

Theorem 1. *Consider the case of recovering a signal $\mathbf{x} \in \mathcal{H}^n$ from m measurements of the form (1) with measurement vectors $\mathbf{a}_i$, $i = 1, 2, \ldots, m$, sampled independently and uniformly from the unit sphere. Let $\text{angle}(\mathbf{x}^0, \hat{\mathbf{x}}) = \arccos\left(\frac{\langle \mathbf{x}^0, \hat{\mathbf{x}} \rangle_{\Re}}{\|\mathbf{x}^0\|_2 \|\hat{\mathbf{x}}\|_2}\right)$ be the angle between the true vector $\mathbf{x}^0$ and the "guess" $\hat{\mathbf{x}}$, and define the constant*

$$\alpha = 1 - \tfrac{2}{\pi}\, \text{angle}(\mathbf{x}^0, \hat{\mathbf{x}})$$

that measures the accuracy of the initialization. If $\mathcal{H} = \mathbb{C}$, then the probability that PhaseMax recovers the true signal $\mathbf{x}^0$ is at least

$$1 - \exp\left(-\frac{(\alpha m - 4n + 1)^2}{2m}\right) \tag{2}$$

whenever $\alpha m > 4n - 1$. In the real case ($\mathcal{H} = \mathbb{R}$), the probability of exact recovery is at least

$$1 - \exp\left(-\frac{(\alpha m - 2n + 1)^2}{2m}\right) \tag{3}$$

whenever $\alpha m > 2n - 1$.

We emphasize that Theorem 1 contains no unspecified constants, and accurately characterizes the empirical behavior of PhaseMax. See Figure 1 for a comparison of the empirical and theoretical success recovery probability in a $n = 500$ dimensional complex system.

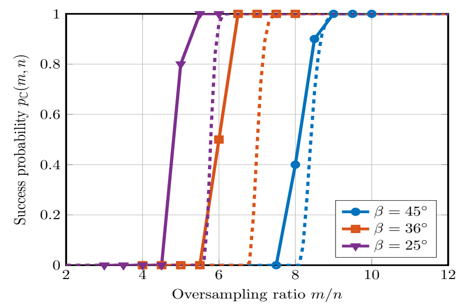

Fig. 1. Comparison between the empirical success probability (solid lines) and our theoretical lower bound (dashed lines) for varying angles between the true signal and the approximation vector.

II. Choosing an Initialization

Several methods exists for choosing an initial guess $\hat{\mathbf{x}}$. We discuss two methods here.

A. Random Guess

A random guess will (with probability 1) have positive correlation with the true signal $\mathbf{x}^0$. As shown in [7], the expected cosine distance between the true signal and a random guess is approximately $(\pi n)^{-1/2}$. Since the accuracy of a random guess decays slowly with increased dimension, a superlinear number of measurement are needed to attain exact reconstruction. In fact, exact reconstruction holds with high probability when $m \geq (\pi n)^{3/2}$.

B. Spectral Initializers

More sophisticated initializers include the (truncated) spectral initializer [2], [5], the Null initializer [9], or the orthogonality-promoting method [6]. For example, given a lower bound $\beta > 0$, the spectral method [2] can guarantee $\text{angle}(\mathbf{x}^0, \hat{\mathbf{x}}) > \beta$ with high probability for $m > c_0 n$, where c_0 is some constant depending only on β. Combining this result with Theorem 1 shows that the truncated spectral initializer enables PhaseMax to succeed with high probability when $m > \max\{4/\alpha, c_1\}n$.

III. Conclusion

Put simply, PhaseMax convexifies phase retrieval problems *without* the pain of lifting. Furthermore, the analysis of PhaseMax rests on a variety of new methods that provide extremely tight reconstruction bounds without any unspecified constants.

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REFERENCES

[1] E. J. Candès, T. Strohmer, and V. Voroninski, “PhaseLift: Exact and stable signal recovery from magnitude measurements via convex programming,” *Commun. Pure Appl. Math.*, vol. 66, no. 8, pp. 1241–1274, 2013.

[2] P. Netrapalli, P. Jain, and S. Sanghavi, “Phase retrieval using alternating minimization,” in *Adv. Neural Inf. Process. Syst.*, 2013, pp. 2796–2804.

[3] P. Schniter and S. Rangan, “Compressive phase retrieval via generalized approximate message passing,” *IEEE Trans. Sig. Process.*, vol. 63, no. 4, pp. 1043–1055, Feb. 2015.

[4] E. J. Candès, X. Li, and M. Soltanolkotabi, “Phase retrieval via Wirtinger flow: Theory and algorithms,” *IEEE Trans. Inf. Theory*, vol. 61, no. 4, pp. 1985–2007, Feb. 2015.

[5] Y. Chen and E. Candès, “Solving random quadratic systems of equations is nearly as easy as solving linear systems,” in *Adv. Neural Inf. Process. Syst.*, 2015, pp. 739–747.

[6] G. Wang, G. B. Giannakis, and Y. C. Eldar, “Solving systems of random quadratic equations via truncated amplitude flow,” *arXiv: 1605.08285*, Jul. 2016.

[7] T. Goldstein and C. Studer, “Phasemax: Convex phase retrieval via basis pursuit,” *arXiv preprint arXiv:1610.07531*, 2016.

[8] S. Bahmani and J. Romberg, “Phase retrieval meets statistical learning theory: A flexible convex relaxation,” *arXiv preprint arXiv:1610.04210*, 2016.

[9] P. Chen, A. Fannjiang, and G.-R. Liu, “Phase retrieval with one or two diffraction patterns by alternating projections of the null vector,” *arXiv:1510.07379*, Apr. 2015.