Sampling the Fourier Transform along Radial Lines

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***Abstract*—This article considers the use of total variation minimization for the recovery of a superposition of point sources from samples of its Fourier transform along radial lines. We present a theoretical result precising the link between the sampling operator and the recoverability of the point sources.**

I. Introduction

The problem of parameter estimation for superpositions of point sources is rooted in applications such as astronomy, NMR (nuclear magnetic resonance) spectroscopy [5], [6] and microscopy [10], [11]. In these applications, the signal of interest can often be modelled as point sources and limitations in the hardware compell one to try to resolve fine details from low frequency data. Physical constraints can sometimes restrict observations to certain angular directions [7], [8], and in the case of NMR spectroscopy, one is required to sample along continuous trajectories such as radial lines.

In this article, we consider this problem under the additional constraint that one can only sample along radial lines in the Fourier domain. Our analysis reveals that the full total variation minimization problem can be solved by considering a sequence of univariate minimization problems which can be tackled via SDP optimization [3]. Our approach is infinite dimensional in the sense that it allows for the recovery of the point sources without resorting to computations on a discrete grid. We show that in dimension d, one can recover the parameters of any superposition of M point sources by sampling its Fourier transform along $d+1$ radial lines for almost all sets of $d+1$ radial lines. Furthermore, the number of samples we require along each line is, up to log factors, linear with M.

II. Exact reconstruction using Fourier Sampling along lines

Let $d \in \mathbb{N}$ with $d \geq 2$ and let $X = \overline{B}(0,1/2) \subset \mathbb{R}^d$ be the centered closed ball with radius $1/2$. We denote by $\mathbb{S}^{d-1}$ the sphere embedded in $\mathbb{R}^d$ and by $\mathcal{M}(X)$ the space of Radon measures with support on X. Given $x \in X$, let δ_x denote the Dirac measure at x. The Fourier transform of $\mu \in \mathcal{M}(X)$ at $\xi \in \mathbb{R}^d$ is defined by

$$\mathcal{F}_d\mu(\xi) = \int_{\mathbb{R}^d} e^{-i2\pi\langle \xi, x\rangle} \mu(\mathrm{d}x).$$

We are interested in the recovery of a discrete measure $\mu_0 = \sum_{j=1}^M a_j \delta x_j$ where $\{x_j\}_{j=1}^M \subset X$ are fixed distinct points and $\{a_j\}_{j=1}^M \subset \mathbb{C}$ are random, given T samples of its Fourier transform along $d+1$ radial lines. More precisely, for $N \in \mathbb{N}$, let $\Gamma \subset [\![-N, N]\!]$ with $T = \mathrm{Card}(\Gamma)$, and let $\Theta \subset \mathbb{S}^{d-1}$ be a set of $d+1$ elements drawn uniformly at random from some set $S \subset \mathbb{S}^{d-1}$ of non-zero spherical measure. Our observation is a vector $y_0 := \Phi\mu_0 \in \mathbb{C}^{T\times(d+1)}$, that is

$$\Phi\mu := (\mathcal{F}_d\mu(k\theta))_{(k,\theta)\in\Gamma\times\Theta} \tag{II.1}$$

$$\text{with} \;\; \mathcal{F}_d\mu_0(\xi) = \sum_{j=1}^M a_j e^{i2\pi\langle\xi, x_j\rangle}. \tag{II.2}$$

Following [1], [2], [3], we consider the solutions to the minimization problem

$$\mathrm{argmin}_{\mu\in\mathcal{M}(X)}\{\|\mu\|_{TV} : \Phi\mu = y_0\} \tag{II.3}$$

where the total variation norm $\|\cdot\|_{TV}$ is defined by

$$\|\mu\|_{TV} = \sup\{\mathrm{Re}\left(\int_X f\mathrm{d}\mu\right) : f \in \mathcal{C}(X,\mathbb{C}), \|f\|_\infty \leq 1\}.$$

In the case of a discrete measure $\mu_0 = \sum_{j=1}^M a_j\delta_{x_j}$, the total variation amounts to $\|\mu_0\|_{TV} = \sum_{j=1}^M |a_j|$.

The following result ensures that the measure μ_0 can be recovered from Fourier samples along $d+1$ radial lines using all the Fourier samples up to frequency N, which is inversely proportional to $\nu_{\min} := \inf_{\theta\in S}\min_{i\neq j} d_{\mathbb{T}}(\langle\theta, x_j\rangle, \langle\theta, x_i\rangle)$ (where $d_{\mathbb{T}}$ is the canonical distance on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$).

Theorem II.1. *Assume that $\nu_{\min} > 0$ and let $N = \lceil 2/\nu_{\min}\rceil$. Then,*

1) *If $\Gamma = [\![-N, N]\!]$, then μ_0 is the unique solution to* (II.3).
2) *If Γ consists of m indices drawn uniformly at random from $[\![-N, N]\!]$, where*

$$m \gtrsim \max\{\log^2(N/\delta), M\log(M/\delta)\log(N/\delta)\},$$

and if $\{sign(a_j)\}_{j=1}^M$ are drawn i.i.d. uniformly at random on $\{z \in \mathbb{C} : |z| = 1\}$, then with probability exceeding $1-(d+1)\delta$, μ_0 is the unique solution to (II.3).

Moreover, μ_0 can be recovered from the resolution of the 1-dimensional problems

$$argmin_{\rho\in\mathcal{M}(\mathbb{T})}\{\|\rho\|_{TV} : (\mathcal{F}_1\rho)_\Gamma = (\mathcal{F}_d\mu_0)_{\{\theta\}\times\Gamma}\} \tag{II.4}$$

The proof relies on the construction of a dual certificate, as a sum of monodimensional dual certificates built from [4] and concentration inequalities.

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REFERENCES

[1] K. Bredies and H. Pikkarainen. Inverse problems in spaces of measures. *ESAIM: Control, Optimisation and Calculus of Variations*, 19(1):190–218, 2013.

[2] De Castro, Yohann and Gamboa, Fabrice *Exact reconstruction using beurling minimal extrapolation*, Journal of Mathematical Analysis and applications 2012, volume 395, number 1, pages 336-354.

[3] E. J. Candès and C. Fernandez-Granda. Towards a mathematical theory of super-resolution. *Communications on Pure and Applied Mathematics*, 67(6):906–956, 2014.

[4] G. Tang, B. N. Bhaskar, P. Shah, and B. Recht. Compressed sensing off the grid. *Information Theory, IEEE Transactions on*, 59(11):7465–7490, 2013.

[5] D. J. Holland, M. J. Bostock, L. F. Gladden, and D. Nietlispach. Fast multidimensional nmr spectroscopy using compressed sensing. *Angewandte Chemie International Edition*, 50(29):6548–6551, 2011.

[6] Y. Wu, C. D'Agostino, D. J. Holland, and L. F. Gladden. In situ study of reaction kinetics using compressed sensing nmr. *Chemical Communications*, 50(91):14137–14140, 2014.

[7] P. N. Stewart, P. G. Tuthill, M. M. Hedman, P. D. Nicholson, and J. P. Lloyd. High-angular-resolution stellar imaging with occultations from the cassini spacecraft–i. observational technique. *Monthly Notices of the Royal Astronomical Society*, 433(3):2286–2293, 2013.

[8] P. N. Stewart, P. G. Tuthill, P. D. Nicholson, M. M. Hedman, and J. P. Lloyd. High angular resolution stellar imaging with occultations from the cassini spacecraft–ii. kronocyclic tomography. *Monthly Notices of the Royal Astronomical Society*, 449(2):1760–1766, 2015.

[9] S. Wenger, S. Darabi, P. Sen, K.-H. Glassmeier, and M. Magnor. Compressed sensing for aperture synthesis imaging. In *2010 IEEE International Conference on Image Processing*, pages 1381–1384. IEEE, 2010.

[10] K. Puschmann and F. Kneer. On super-resolution in astronomical imaging. *Astronomy & Astrophysics*, 436(1):373–378, 2005.

[11] V. Studer, J. Bobin, M. Chahid, H. S. Mousavi, E. Candes, and M. Dahan. Compressive fluorescence microscopy for biological and hyperspectral imaging. *Proceedings of the National Academy of Sciences*, 109(26):E1679–E1687, 2012.