Sparse Super-Resolution from Laplace Measurements

Quentin Denoyelle
Univ. Paris-Dauphine
INRIA Mokaplan
denoyelle@ceremade.dauphine.fr

Emmanuel Soubies
Biomedical Imaging Group
EPFL
emmanuel.soubies@epfl.ch

Vincent Duval
INRIA Mokaplan
Univ. Paris-Dauphine
vincent.duval@inria.fr

Gabriel Peyré
CNRS & DMA
ENS
gabriel.peyre@ens.fr

***Abstract*—We propose a theoretical analysis of the super-resolution performance of the BLASSO "off-the-grid" recovery method from Laplace transform measurements. This transform is not translation invariant, thus requiring the use of theoretical and algorithmic tools that go beyond traditional deconvolution-based methods. We show that the BLASSO offers a stable and computationally tractable super-resolution of positive spikes. In particular, when the signal-to-noise ratio is of the order of $1/t^{2N-1}$ (where t is the spacing between the N spikes to recover), the BLASSO program outputs the correct number of spikes. This result suggests that the BLASSO should be a method of choice to tackle challenging Laplace inversions, which are at the heart of recently proposed fluorescence imaging methods.**

I. Introduction and Previous Works

a) Sparse Spikes Super-Resolution: it aims at recovering highly localised features from noisy low-resolution observations. The sparse unknown data to recover is modelled as a sum of Dirac masses $m_0 \stackrel{\text{def.}}{=} \sum_{i=1}^N a_i \delta_{x_i}$ with amplitudes $a_i > 0$ and positions $x_i \in X$ on some domain X (for instance an interval of $\mathbb{R}$, or some higher-dimensional set). The noisy low-resolution observations are defined as $y = \Phi m_0 + w \in \mathcal{H}$ where $w \in \mathcal{H}$ is an additive noise. The linear map Φ is the imaging operator, and it accounts for the measurement device (for instance a camera or a medical imaging device). This operator is defined on the space $\mathcal{M}(X)$ of Radon measures on X (which includes in particular sums of Dirac masses) and takes values in some Hilbert space $\mathcal{H}$ (typically $\mathbb{R}^P$ or $\mathbb{C}^P$ for a finite number of measurements, or a functional space for continuous measurements). Such a linear map is conveniently written as $\forall m \in \mathcal{M}(X), \quad \Phi m = \int_X \varphi(x) \mathrm{d}m(x)$, where $\varphi : X \to \mathcal{H}$ is a smooth kernel function.

Although this is a simple setup, it covers a surprisingly large class of problems of practical interest, such as decoding neural spikes in electrophysiology [9] or single-molecule fluorescence imaging [12]. Many applications consider that the operator Φ is translation-invariant (corresponding to a deconvolution problem). However, there is a growing interest in more involved non-translation-invariant operators. Among these, one of the simplest is the Laplace transform, that we study in this article. This operator is for instance at the heart of the multi-angle TIRF imaging technic, allowing to reach an axial (depth) super-resolution of the observed molecules by inverting a 1-D Laplace transform, see for instance [2].

b) BLASSO Off-the-Grid Recovery: Following several recent works, we consider the BLASSO convex optimization problem over the space $\mathcal{M}(X)$

$$\min_{m \in \mathcal{M}(X)} \frac{1}{2}\|\Phi m - y\|_{\mathcal{H}}^2 + \lambda |m|(X), \qquad (\mathcal{P}_\lambda(y))$$

where $\lambda > 0$ is the regularization parameter and $|m|(X)$ is the total variation of the measure m (not to be mistaken for the total variation of the gradient, commonly used in image processing), that extends the discrete ℓ^1 norm of vectors to measures

$$|m|(X) = \sup \left\{ \int_X \psi \mathrm{d}m; \ \psi \in C(X), \|\psi\|_{\mathrm{L}^\infty(X)} \leq 1 \right\}.$$

Problem $(\mathcal{P}_\lambda(y))$ does not introduce any discretization grid, and is thus both easier to analyze theoretically and leads to more precise super-resolution results. We refer the reader to [3], [4], [5], [6], [7], [8], [10], [11] for more details about the BLASSO program, its properties, and computational algorithms.

c) Recovery Properties: We consider in the following the 1-D case $X = \mathbb{R}$. To study the super-resolution capability of $(\mathcal{P}_\lambda(y))$, we consider the case where all the Dirac masses cluster near some central position $\bar{x} \in X$, i.e. $x_i = \bar{x} + t z_i$ for some arbitrary directions z_i and a small $t > 0$. It is possible to extend the theory to multiple clustering points.

It is shown in [11] that, in the limit of small t, the recovery performance of $(\mathcal{P}_\lambda(y))$ is governed by the following smooth function $\eta_W : \mathbb{R} \to \mathbb{R}$ (which only depends on the clustering position $\bar{x}$).

Definition 1 (Asymptotic pre-certificate)**.** *Let $\eta_W = \Phi^* p_W$ where*

$$p_W \stackrel{\text{def.}}{=} \operatorname*{argmin}_{p \in \mathcal{H}} \left\{ \|p\|_{\mathcal{H}}; (\Phi^* p)^{(k)}(\bar{x}) = \delta_0^k, 0 \leq k < 2N \right\}.$$

This function has all its derivatives up to order $2N-1$ that vanish at $\bar{x}$ and can be evaluated numerically by solving a $2N \times 2N$ linear system [11].

η_W is said to be "non-degenerate" if $|\eta_W(x)| < 1$ for $x \neq \bar{x}$ and if $\eta_W^{(2N)}(\bar{x}) \neq 0$. We showed in [11] that if η_W is non-degenerate and if $(\frac{w}{t^{2N-1}}, \frac{\lambda}{t^{2N-1}}, \frac{w}{\lambda}) = O(1)$, then the measure solving $(\mathcal{P}_\lambda(y))$ is unique and is composed of exactly N Diracs located close to the initial ones. This signal-to-noise condition matches (up to a constant) the Cramer-Rao bounds (when the noise is assumed to be Gaussian distributed) [1], suggesting the good practical behavior of the method.

II. Contributions

We study two variants of the Laplace transform, defined on $X = [a, +\infty[$ for some $a > 0$ (the input position 0 needs to be avoided to ensure a finite-energy transform) and $\mathcal{H} = L^2(\mathbb{R}^+)$ (endowed with the classical scalar product). These transforms are defined through their kernels φ, which are

$$\varphi(x) : h \in (0, +\infty) \longmapsto \begin{cases} e^{-hx} & \text{(un-normalized)}, \\ \sqrt{2x} e^{-hx} & \text{(normalized)}. \end{cases}$$

In the normalized case, $\|\varphi(x)\|_{\mathcal{H}} = 1$ (which is the usual normalization constraint for sparse recovery methods), so that this is the transform that should be preferred in practice to avoid introducing bias in the recovered positions. Our main contribution is the following result, which shows that for the Laplace transform, solving $(\mathcal{P}_\lambda(y))$ leads to a stable super-resolution for positive measures.

Theorem 1. *η_W is non-degenerate, and $\forall x > a$,*

$$\eta_W(x) = \begin{cases} 1 - \left(\frac{x-\bar{x}}{x+\bar{x}}\right)^{2N} & \textit{(un-normalized)}, \\ \frac{2\sqrt{x\bar{x}}}{x+\bar{x}} \sum_{k=0}^{2N-1} \frac{1 \cdot 3 \cdots (2k-3)}{2^k k!} \left(\frac{x-\bar{x}}{x+\bar{x}}\right)^{2k} & \textit{(normalized)}. \end{cases}$$

Figure 1 illustrates the behavior of η_W for different values of N. Future works include the use of the conditional gradient algorithm [3] to solve numerically large-scale problems for MA-TIRF axial single molecule localization [2], to highlight the practical relevance of our theoretical findings.

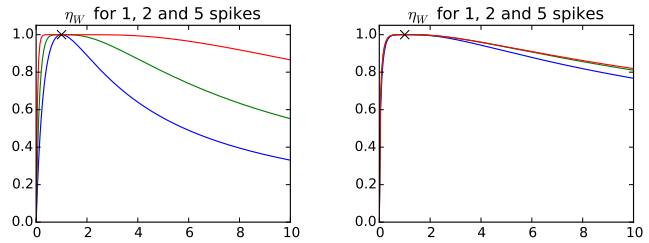

Fig. 1. η_W for the Laplace Transform in the un-normalized (left) and normalized (right) case, for an increasing number of spikes.

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