Variable Splitting and Cycle Spinning for Sparse Signal Recovery

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***Abstract*—We propose a variable splitting approach for sparse recovery from incomplete Fourier data, which significantly improves conventional wavelet-based compressed sensing/reconstruction, offers the benefits of Shift-invariant Wavelet Transform (SWT), and overcomes the high redundancy factor of SWT. Our method recovers sparse Discrete Wavelet Transform (DWT) coefficients of translated version of the signal in parallel, while enforces consistency between the translated signals via solving the problem in an ADMM formulation. The experiments demonstrate that 4 shifts are sufficient to achieve reconstruction accuracy as high as reconstruction using SWT, hence, significantly reducing the computational cost and redundancy factor of SWT frame.**

I. Introduction

Effective signal reconstruction from limited number of Fourier measurements is key to several applications such as medical imaging (e.g., tomography, MRI), seismic and astronomical imaging, and radar. Sparse representation in wavelet domain has been extensively used in compressed sensing research, owing to effectiveness of wavelet bases in sparsifying natural images. While DWT is efficient in sparse representation of signals, it introduces pseudo-Gibbs artifacts to the reconstructed image, mostly attributed to its *shift-variant* nature [4]. Ideally one replaces DWT with SWT to achieve translation-invariance; nevertheless, the high redundancy factor of SWT limits its application in practical settings.

Coifman and Donoho [1] introduced the concept of cycle spinning for wavelet denoising, in which several shifted versions of the signal are denoised separately and linearly averaged to obtain the final (denoised) signal. While cycle spinning is efficient in denoising, since each shifted signal is tackled separately, there is no guarantee that resulting signals are consistent (agree with each other and represent the same signal). Kamilov et al., [2] discussed the inconsistency issue, proposed a denoising approach for consistent cycle spinning with 1-level Haar transform, and established its equivalence to total variation minimization.

In contrast, we propose an approach for sparse signal reconstruction from limited Fourier data based on variable splitting that significantly outperforms conventional wavelet-based reconstruction, guarantees consistency of shifted signal, and overcomes the high redundancy factor of SWT.

II. Cycle Spinning for Partial Fourier Reconstruction

When a signal in $\mathbb{R}^N$ is sparse (or compressible) in a dictionary (or a basis), $\mathbf{D}$, the recovery problem from partial Fourier measurements can be written as:

$$\operatorname*{arg\,min}_{\mathbf{u}\in\mathbb{R}^N} \quad ||\mathbf{D}\mathbf{u}||_1 + \frac{\mu}{2}||\boldsymbol{\mathcal{F}}_\Omega \mathbf{u} - \mathbf{f}||_2^2, \tag{1}$$

where $\boldsymbol{\mathcal{F}}_\Omega$ is the Fourier matrix restricted to set of frequencies Ω, and $\mathbf{f} \in \mathbb{C}^n$ with $(n \ll N)$ denotes the given data. When the dictionary $\mathbf{D}$ is overcomplete, the problem size grows very large due to high redundancy factor of the transform. For example, SWT with Haar wavelet and p levels of decomposition has a redundancy factor of $3p+1$ (e.g., for a 256×256 image at full decomposition, redundancy factor is 25). In this paper, we propose to reformulate (1) as k sparse approximation problems for obtaining shifted versions of the signal:

$$\operatorname*{arg\,min}_{\mathbf{u}\in\mathbb{R}^N} \quad ||\boldsymbol{\Psi}\boldsymbol{S}_i\mathbf{u}||_1 + \frac{\mu}{2}||\boldsymbol{\mathcal{F}}_\Omega \mathbf{u} - \mathbf{f}||_2^2 \qquad i = 1\dots k \tag{2}$$

where $\boldsymbol{S}_i$ is the i$^{\text{th}}$ shift operator and $\boldsymbol{\Psi}$ is the DWT matrix. We solve the k separate problem in (2) *simultaneously*:

$$\operatorname*{arg\,min}_{\mathbf{u}\in\mathbb{R}^N} \quad \frac{\mu k}{2}||\boldsymbol{\mathcal{F}}_\Omega \mathbf{u} - \mathbf{f}||_2^2 + ||\boldsymbol{\Psi}\boldsymbol{S}_1\mathbf{u}||_1 + \dots + ||\boldsymbol{\Psi}\boldsymbol{S}_k\mathbf{u}||_1 \tag{3}$$

This can be solved efficiently using Bregman splitting [3] (as described in Algorithm 1):

$$\begin{aligned}\operatorname*{arg\,min}_{\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_k,\mathbf{u}\in\mathbb{R}^N} \quad & \frac{\mu k}{2}||\boldsymbol{\mathcal{F}}_\Omega \mathbf{u} - \mathbf{f}||_2^2 + \sum_{i=1}^{k}||\boldsymbol{\theta}_i||_1 + \\ & + \frac{\lambda}{2}\sum_{i=1}^{k}||\boldsymbol{\theta}_i - \boldsymbol{\Psi}\boldsymbol{S}_i\mathbf{u} - \mathbf{b}_{\boldsymbol{\theta}_i}||_2^2,\end{aligned} \tag{4}$$

where $\mathbf{b}_{\boldsymbol{\theta}_i}$ $(i = 1,\dots,k)$ denotes the vector of Lagrange multipliers.

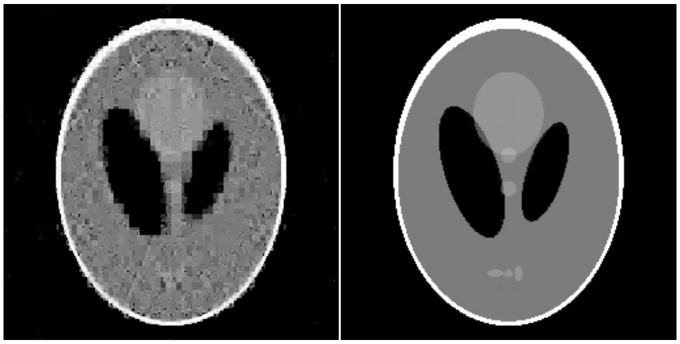
Algorithm 1 Variable Splitting with k-Shift Cycle Spinning

Initialize: $\mathbf{f}^0 = \mathbf{f}, \boldsymbol{\theta}_1^0,\dots,\boldsymbol{\theta}_k^0 = \mathbf{0}, \mathbf{b}_{\boldsymbol{\theta}_1}^0,\dots,\mathbf{b}_{\boldsymbol{\theta}_k}^0 = \mathbf{0}, t = 0$
while $||\mathbf{u}^{t+1} - \mathbf{u}^t||_2 > tol$
  for $l = 1$ to L
    $\mathbf{u}^{t+1} = \operatorname*{arg\,min}_{\mathbf{u}\in\mathbb{R}^N} \quad \frac{\mu k}{2}||\boldsymbol{\mathcal{F}}_\Omega\mathbf{u} - \mathbf{f}^t||_2^2 + \frac{\lambda}{2}\sum_{i=1}^{k}||\boldsymbol{\theta}_i^t - \boldsymbol{\Psi}\boldsymbol{S}_i\mathbf{u} - \mathbf{b}_{\boldsymbol{\theta}_i}^t||_2^2$
    $\boldsymbol{\theta}_i^{t+1} = \operatorname*{arg\,min}_{\boldsymbol{\theta}_i\in\mathbb{R}^N} \quad ||\boldsymbol{\theta}_i||_1 + \frac{\lambda}{2}||\boldsymbol{\theta}_i - \boldsymbol{\Psi}\boldsymbol{S}_i\mathbf{u}^{t+1} - \mathbf{b}_{\boldsymbol{\theta}_i}^t||_2^2 \quad i = 1,\dots,k$
    $\mathbf{b}_{\boldsymbol{\theta}_i}^{t+1} = \mathbf{b}_{\boldsymbol{\theta}_i}^t + (\boldsymbol{\Psi}\boldsymbol{S}_i\mathbf{u}^{t+1} - \boldsymbol{\theta}_i^{t+1}) \quad i = 1,\dots,k$
  $\mathbf{f}^{t+1} = \mathbf{f}^t + \mathbf{f}^0 - \boldsymbol{\mathcal{F}}_\Omega\mathbf{u}^{t+1}$
  $t \leftarrow t + 1$
return $\mathbf{u}^t$

The update step for $\mathbf{u}$ is in quadratic form, hence, has a closed form solution, while updates for $\boldsymbol{\theta}_i$'s are performed via shrinkage. We note that we can perform all k updates for $\boldsymbol{\theta}_i$ and $\mathbf{b}_{\boldsymbol{\theta}_i}$ in parallel, which allows for adding more shifts without sacrificing the efficiency of the approach.

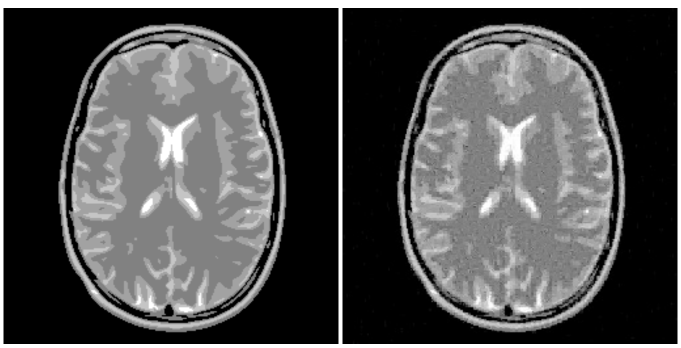
III. Results and Discussion

Our experiments suggest that the high redundancy factor of SWT can be overcome via the proposed method, where few number of shifts suffice to achieve similar accuracy. Fig. 1 shows that our approach achieves exact reconstruction for Shepp-Logan phantom at sampling rate as low as 3.06%, while reconstruction with DWT fails to recover the image. Fig. 2 compares reconstruction with DWT against SWT and the proposed method with 4 shifts. While the proposed method offers similar accuracy as SWT, it reduces the reconstruction time by a factor of 22. A more thorough investigation of effectiveness of this approach is presented in [5].



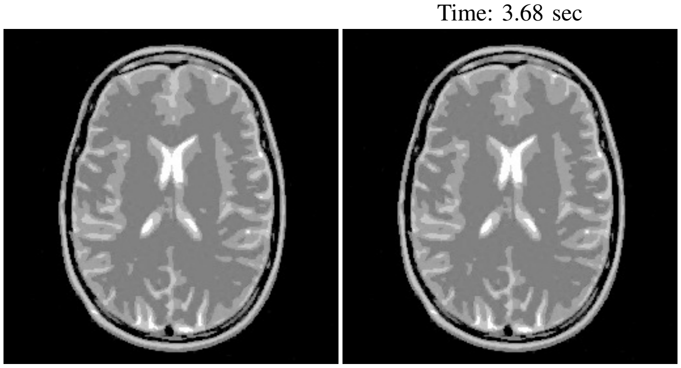
(a) SNR: 11.24 dB (b) SNR: 37.24 dB

Fig. 1. Reconstruction of Shepp-Logan phantom of size 256×256 from 3.06% of Fourier data (radial sampling in frequency domain): (a) DWT-based reconstruction, (b) Reconstruction via variable splitting with 4 shifts. The proposed approach achieves exact reconstruction at sampling rate as low as 3.06%, while the ADMM formulation allows for distributed computation of multiple shifts.



(a) Ground Truth

(b) DWT
SNR: 15.95 dB
Time: 3.68 sec



(c) SWT
SNR: 22.59 dB
Time: 212.68 sec

(d) Variable Splitting with 4 shifts
SNR: 22.53 dB
Time: 9.48 sec

Fig. 2. Reconstruction of brain phantom of size 256×256 from 9.24% of Fourier data: (a) the ground truth image (b) DWT-based reconstruction, (c) SWT-based reconstruction (redundancy factor 25). (d) Reconstruction via variable splitting with 4 shifts, shift set={(0,0),(1,1),(2,2),(3,3)} The proposed approach achieves reconstruction accuracy similar to reconstruction with sparsity in SWT, using only 4 shifts, which reduces the reconstruction time from 212.68 seconds down to 9.48 seconds (approximately 22 times reduction in computational cost)

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