Uniform recovery guarantees for Hadamard sampling and wavelet reconstruction

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***Abstract*—In most real-world applications of compressed sensing uniform random sampling is suboptimal [2, 5, 12, 21, 25], however structured sampling is an effective alternative. In order to obtain satisfactory signal reconstructions in such applications one need to incorporate both the signal structure, and the local coherence structure of the change of basis matrix in the choice of sampling patterns. In this text we will estimate the local coherences in a change of basis matrix between Hadamard samples and Daubechies wavelets. These estimates are then combined with newly obtained uniform recovery guarantees, to create concrete guarantees for Hadamard sampling combined with Daubechies wavelets.**

I. INTRODUCTION

Sparsity alone is typically too general an assumption for many applications in compressive imaging [1, 4, 14, 16, 23]. Standard examples range from Magnetic Resonance Imaging (MRI) [15, 17, 18], surface scattering [13], Computerized Tomography (CT), all of which employ Fourier samples, to fluorescence microscopy [24] and lensless imaging [26], using binary sampling. For these applications, when using X-lets, a natural alternative is to consider a sparsity in levels model [6, 8, 20, 22]. The required number of samples within each level will then depend on the local sparsity in the level and the coherence structure between the sampling operator and the sparsifying operator [3]. In the case of Fourier samples and wavelet reconstruction this is reasonably well understood, however for binary sampling, where Hadamard sampling is one of the most natural sampling operators, this is not the case.

II. MAIN RESULT

Let $f \in L^2([0,1))$ be the signal we are trying to recover from the Hadamard samples $x_n = \langle f, w_n \rangle$, where $w_n \colon [0,1) \to \{+1,-1\}$ is the Walsh function. It is this function which generates the rows in a Hadamard matrix.

Definition II.1 (Walsh function)**.** Let $n = n_1 2^0 + n_2 2^1 + \cdots$, with $n_i \in \{0,1\}$ be the dyadic expansion of $n \in \mathbb{N}$. Similarly let $x = x_1 2^{-1} + x_2 2^{-2} + \cdots$ with $x_i \in \{0,1\}$ be the dyadic expansion of $x \in [0,1)$. The *sequency ordered Walsh function* is

$$w_n(x) := (-1)^{\sum_{i=1}^{\infty}(n_i + n_{i+1})x_i}$$

One of the basic problems in compressive sensing [7, 11] is to design a sampling pattern $\Omega \subset \{1,\ldots,N_r\}$ with $|\Omega| = m$ such that by solving

$$\text{minimize}_{z\in\mathbb{C}^M} ||z||_1 \quad \text{subject to} \quad ||Az - y||_2 \leqslant \eta_1 + \eta_2, \qquad (1)$$

where $A = P_\Omega U P_M$, one gets a good approximation to $x = (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ originating from the noisy inverse problem $y = P_\Omega U x + q$. Here $U \in \mathcal{B}(\ell^2(\mathbb{N}))$, $||q||_2 \leqslant \eta_1$, $||P_{N_r} U P_M^{\perp} x|| \leqslant \eta_2$ and P_Ω is the projection onto span $\{e_j : j \in \Omega\}$, where e_j is the canonical basis, $P_M := P_{\{1,\ldots,M\}}$ and $P_M^{\perp} := P_{\{M+1,\ldots\}}$ In our setup we consider the change of basis matrix $U_{nz} = \langle w_n, \varphi_z \rangle$ where $\{\varphi_z\}_{z\in\mathbb{N}}$ is a Daubechies wavelet basis [9, 10, 19] ordered such that any wavelet at a lower scale precede wavelets at higher scales. We partition the sampling pattern Ω into r levels $\mathbf{N} = [N_1,\ldots,N_r] \in \mathbb{N}^r, N_0 = 0$. From each of these levels we draw a set $\Omega_k \subset \{N_{k-1}+1,\ldots,N_k\}$ of size $|\Omega_k| = m_k$ uniformly at random. The m_k samples from the k'th level are then obtained as $P_{\Omega_k} x$.

Further we assume that the sparsity structure of the signal in the wavelet domain, i.e. $\tilde{y} = P_{N_r} U x$, can be partitioned into $\mathbf{M} = [M_1,\ldots,M_r] \in \mathbb{N}^r, M_0 = 0$, sparsity levels, each of which contains $0 < s_k \leqslant M_k - M_{k-1}$ non-zero coefficients. We call such a vector $(\mathbf{M}, \boldsymbol{s})$-sparse, $\boldsymbol{s} = [s_1,\ldots,s_r]$, and let $\Sigma_{\boldsymbol{s},\mathbf{M}}$ denote the collection of all such vectors. In [16] Li and Adcock derived a general uniform recovery guarantee for (1), whenever U was an isometry on $\mathbb{C}^N$. We have extended this to isometries U on $\ell^2(\mathbb{N})$ solving (1) with

$$A = \begin{bmatrix} 1/\sqrt{p_1} P_{\Omega_1} U P_M \\ \vdots \\ 1/\sqrt{p_r} P_{\Omega_r} U P_M \end{bmatrix} \in \mathbb{C}^{m\times M}, \qquad p_k = \frac{m_k}{N_k - N_{k-1}}. \qquad (2)$$

This result relies on the ability to estimate the local coherences of U. That is

$$\mu_{k,l} = \max\{|U_{ij}|^2 : N_{k-1} < i \leqslant N_k, M_{k-1} < j \leqslant M_k\}.$$

We have estimated these local coherences for the U described above. This leads to the following theorem.

Theorem II.2. *Let $U_{nz} = \langle w_n, \varphi_z \rangle$ be an isometry on $\ell^2(\mathbb{N})$, where φ_z is a Daubechies wavelet with $\nu > 2$ vanishing moments. Let $\mathbf{M} = [2^{J_0+1},\ldots,2^{J_0+r}]$ be sparsity levels (corresponding to wavelet scales), and $\mathbf{N} = [2^{J_0+1},\ldots,2^{J_0+r-1},2^{J_0+r+p}]$, $p \geqslant 0$, be sampling levels. Suppose $P_{M_r} U^* P_{N_r} U P_{M_r} = G^*G$ is invertible. Let $\kappa(G)$ be the condition number of G, $0 < \epsilon < 1$ and $\rho = \max_{i,j} s_i/s_j$. Suppose that the local sampling densities m_k satisfy*

$$m_k \geqslant C\kappa(G)^2 r\rho \left(\sum_{t=1}^{r} 2^{-|k-t|} s_t\right) \cdot \left(r\log(2m)\log(2N_r)\log^2(2s) + \log(\epsilon^{-1})\right)$$

where $m = m_1 + \cdots + m_r$, $s = s_1 + \cdots + s_r$, and C is a constant independent of all relevant parameters. Then with probability at least $1-\epsilon$, any minimizer $\hat{x}$ of (1) *with A given by* (2) *is bounded by*

$$||\hat{x} - x||_1 \lesssim \sigma_{\boldsymbol{s},\mathbf{M}}(x) + ||G^{-1}||_2(\eta_1 + \eta_2)\sqrt{s}$$

where $\sigma_{\boldsymbol{s},\mathbf{M}}(x) = \inf\{||x - z||_1 : z \in \Sigma_{\boldsymbol{s},\mathbf{M}}\}$ and η_1, η_2 are as in (1).

This gives concrete estimates for the local sampling densities m_k for signal recovery up to a best $(\boldsymbol{s},\mathbf{M})$-term approximation using Hadamard sampling.

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