Learning Convolutional Proximal Filters

Ulugbek S. Kamilov, Hassan Mansour, and Dehong Liu
Mitsubishi Electric Research Laboratories (MERL)
201 Broadway, Cambridge, MA, 02139, USA
Email: kamilov@merl.com, mansour@merl.com, and liudh@merl.com

Abstract—In the past decade, sparsity-driven methods have led to substantial improvements in the capabilities of numerous imaging systems. While traditionally such methods relied on analytical models of sparsity, such as total variation (TV) or wavelet regularization, recent methods are increasingly based on data-driven models such as dictionary-learning or convolutional neural networks (CNN). In this work, we propose a new trainable model based on the proximal operator for TV. By interpreting the popular fast iterative shrinkage/thresholding algorithm (FISTA) as a CNN, we train the filters of the algorithm to minimize the error over a training data-set. Experiments on image denoising show that by training the filters, one can substantially boost the performance of the algorithm and make it competitive with other state-of-the-art methods.

I. INTRODUCTION

We consider an inverse imaging problem \( y = Hx + e \), where the goal is to recover the unknown image \( x \in \mathbb{R}^N \) from the noisy measurements \( y \in \mathbb{R}^M \). The matrix \( H \in \mathbb{R}^{M \times N} \) is known and models the response of the acquisition device, while the vector \( e \in \mathbb{R}^N \) represents the unknown noise in the measurements.

Practical imaging inverse problems are often ill-posed [1]. A standard approach for solving such problems is the regularized least-squares estimator

\[
\hat{x} = \arg \min_{x \in \mathbb{R}^N} \left\{ \frac{1}{2} \| y - Hx \|^2_2 + R(x) \right\},
\]

where \( R \) is a regularizer promoting solutions with desirable properties. One of the most popular regularizers for images is the total variation (TV) [2], defined as \( R(x) = \tau \| D_x \|_1 \), where \( \tau > 0 \) is a parameter that controls the strength of the regularization, and \( D : \mathbb{R}^N \rightarrow \mathbb{R}^{N \times K} \) is the discrete gradient operator. The gradient can be represented with \( K \) separate filters, \( D \triangleq (D_1, \ldots, D_K) \), computing finite-differences along each dimension of the image.

Two common methods for solving the TV regularized problem (1) are fast iterative shrinkage/thresholding algorithms (FISTA) [3] and alternating direction method of multipliers (ADMM) [4]. These algorithms are among the methods of choice for solving large-scale imaging problems due to their ability to handle the non-smoothness of TV and their low-computational complexity. Both FISTA and ADMM typically combine the operations with the measurement matrix with applications of the proximal operator

\[
\text{prox}_\tau R(y) \triangleq \arg \min_{x \in \mathbb{R}^N} \left\{ \frac{1}{2} \| x - y \|^2_2 + \tau R(x) \right\}.
\]

Beck and Teboulle [3] have proposed an efficient dual domain FISTA for computing TV proximal

\[
\begin{align}
\ell_t^t &= g_{t-1}^t + ((q_t - 1)/q_t)g_{t-2}^t \label{eq:2a} \\
\delta_t^t &= s_{t-1}^t - \gamma \tau D (\tau D^T s_{t-1} - y) \label{eq:2b} \\
g_{t}^t &= \mathcal{P}_\infty(x_t) \label{eq:2c}
\end{align}
\]

with \( q_0 = 1 \) and \( g_{-1}^0 = g_{-2}^0 \in \mathbb{R}^{N \times K} \). Here, \( \mathcal{P}_\infty \) denotes a component-wise projection operator onto a unit \( \ell_\infty \)-norm ball, \( \gamma = 1/L \) with \( L = \tau^2 \lambda_{\max}(D^T D) \) is a step-size, and \( \{q_t\}_{t \in \mathbb{N}} \) are relaxation parameters. For a fixed \( q_t = 1 \), the guaranteed global convergence speed of the algorithm is \( O(1/t) \); however, the choice \( q_t = 1/(1 + \sqrt{1 + 4q_{t-1}}) \) leads to a faster \( O(1/t^2) \) convergence [3]. The final denoised image after \( T \) iterations of (3) is obtained as \( x^T = y - \tau D^T g^T \).

II. MAIN RESULTS

Our goal is to obtain a trainable variant of (3) by replacing the finite-difference TV with a CNN, where a component of the formula is convolutional neural network (CNN) of a particular structure with \( T \times K \) filters \( D_t ^\triangleq (D_{t1}, \ldots, D_{tK}) \) that are learned from a set of \( L \) training examples \( \{x_t, y_t\} \in \mathbb{R}^{N \times K} \). The filters can be optimized by minimizing the error

\[
\hat{\theta} = \arg \min_{\theta \in \Theta} \left\{ \frac{1}{L} \sum_{l=1}^{L} E_l(\theta) \right\} \quad \text{with} \quad E_l(\theta) \triangleq \|x - \hat{x}(y; \theta)\|^2_2 \quad (4)
\]

over the training set, where \( \theta = (D_{l1}, \ldots, D_{lK}) \in \Theta \) denotes the set of desirable filters. The problem of image denoising, end-to-end optimization can be performed with the error backpropagation algorithm [5] that produces

\[
\left[ \nabla E_l(\theta) \right]_{[k]} = \begin{cases} q_{tk} + \tau (g_{tk}^T \bullet (x - \hat{x})) & \text{for } t = T \\ q_{tk} & \text{for } 1 \leq t \leq T - 1, \end{cases}
\]

using the following iteration for \( t = T, T - 1, \ldots, 1 \),

\[
\begin{align}
v_t^{[-1]} &= \text{diag} \left( \{\mathcal{P}_\infty(z_t^{[-1]})\}^T \right) v_t^{[-1]} \quad (5a) \\
b_t^{[-1]} &= \mu_s b_t^{[-1]} + (1 - \mu_s) b_t^{[-1]} \quad (5b) \\
s_t^{[-1]} &= \gamma \tau [v_t^{[-1]} \bullet (y - \tau D^T s_t^{[-1]}) - \tau (s_t^{[-1]} \bullet (D_t^T v_t^{[-1]})))] 
\quad \text{for } t = T, T - 1, \ldots, 1. \quad (5c)
\end{align}
\]

where \( \bullet \) denotes filtering, \( \mu_s = 1 - (1 - q_{t-1})/q_t \), \( b_t^{[-1]} = 0 \), and \( r_t^{[-1]} = \tau D^T (x - \hat{x}) \). The parameters are update iteratively with the standard stochastic gradient method as \( \theta^{\tau} = \theta - \alpha \nabla E_l(\theta) \).

We applied our method to image denoising by training \( T = 10 \) iterations of the algorithm with \( K = 9 \) iteration dependent kernels of size \( 6 \times 6 \) pixels. For training, we used 400 images from the Berkeley dataset [6] cropped to \( 192 \times 192 \) pixels. We evaluated the algorithm on 68 separate test images from the dataset and compared the results with three popular denoising algorithms (see Table I and Fig. 2–3). Our basic MATLAB implementation takes 0.69 and 3.27 seconds on images of \( 256 \times 256 \) and \( 512 \times 512 \) pixels, respectively, on an Apple iMac with a 4 GHz Intel Core i7 processor. We observe that our simple extension of TV significantly boosts the performance of the algorithm and makes it competitive with state-of-the-art denoising algorithms. The algorithm can be easily incorporated into FISTA and ADMM for solving more general inverse problems. Future work will address such extensions and further improve the performance by code optimization and considering more kernels. More generally, our work contributes to the recent efforts to boost the performance of imaging algorithms by incorporating latest ideas from deep learning [7]–[13].
Fig. 1. A schematic representation of the trainable variant of (3) with adaptable parameters, $W_t \triangleq \mathbf{I} - \gamma \tau D_t \mathbf{D}_t^T$ and $b_t \triangleq \gamma \tau D_t \mathbf{y}$, marked in blue. (a) The algorithm for $T = 3$ iterations with $\theta_t \triangleq D_t$. (b) The schematic view of a single iteration where $\mu_t = 1 - (1 - q_{t-1})/q_t$. (c) The plot of the scalar nonlinearity $\mathcal{P}_\infty$.

### Table I

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References
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